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A random walk approach to Stochastic Calculus

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Resum *(CAT)*
L'objectiu d'aquest treball és presentar una introducció al càlcul estocàstic. En la primera part parlem del moviment brownià, el qual veurem que es pot pensar com a límit de passeigs aleatoris amb l'ajut del principi d'invariància de Donsker.
A continuació, presentem de manera heurística les equacions diferencials estocàstiques i veiem com es poden definir de manera rigorosa amb l'ajut de la integral estocàstica. Finalment, parlem d'existència i unicitat de solucions d'aquestes equacions i tractem un cas senzill com és el de l'equació de Langevin.

Abstract *(ENG)*
The aim of this work is to provide an introduction to the subject of Stochastic Calculus. In the first part we talk about the Brownian motion, which we will see that it can be thought as a limit of random walks via Donsker's Invariance Principle. Next, we heuristically present the stochastic differential equations and see how they can be rigorously defined with the help of the stochastic integral. Finally, we discuss the matter of existence and uniqueness of solutions to such equations and solve a rather simple case like the Langevin equation.

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1. Introduction

In subjects like Thermodynamics and Statistical Mechanics, in several occasions one gives a stochastic approach of a problem even though it can be treated in a deterministic way because it usually leads to simple and less tedious formulations and computations. For instance, if we want to study the motion of a particle in a fluid, it is much more simpler to think that the object moves randomly due to the several collisions that are happening in the system, rather than considering each interaction individually and try to force brute Newton's equations into the system.

This approach, which seems promising, comes with a couple of drawbacks. The first one is that we have to give up on trying to determine the exact trajectory of the particle, since, even if the initial conditions are the same, different identical particles might describe different sample paths.

The other drawback, which we shall focus our attention on, is that these kind of formulations usually lead to equations like the Langevin equation:

$$\frac{d\dot{X}_t}{dt} = -\mu \dot{X}_t + \dot{F}_t, \quad \dot{X}_t = \frac{dX_t}{dt}, \tag{1}$$

where μ is some positive real constant, X_t is the position of the particle at time $t \geq 0$ and $\dot{F}_t$ is a random perturbation that evolves with time and satisfies some conditions like $\mathbb{E}[\dot{F}_t] = 0$ and $\mathbb{E}[\dot{F}_t \dot{F}_s] = \Gamma \delta(t-s)$ (being $\mathbb{E}$ the expectation operator, Γ some positive real constant and δ the Dirac delta). Many physicists say that the process $X = \{X_t : t \geq 0\}$, where X_t is the position described by the latter equation, is a Brownian motion. However, it is very well-known that the sample paths of such process are nowhere differentiable in closed intervals with probability one, meaning that expressions like $\dot{X}_t$ (and higher order derivatives) make no sense when they are considered pathwise, so we must find a way to define such objects (derivatives of functions which are not differentiable in the usual sense) in order to be able to give a rigorous definition of equations like (1). Before doing so, we first need to define what is a Brownian motion. More particularly, we must check that we can define a mathematical object satisfying the properties that a process like the one described by (1) should satisfy.

2. Construction of the Brownian motion

When one asks what is a Brownian motion to someone who is not familiar with the subject of stochastic processes, the usual answer is that it is the random movement of a particle suspended in some medium (a liquid or the air, for instance). In some other cases, the answer is that it is the movement described by a particle that makes small, random displacements which behave similarly, even though they seem uncorrelated no matter what the position of the object is.

But all these features are already satisfied by a random walk whose jumps are "small" (for instance, of finite variance). Indeed, recall that a random walk is a process $S = \{S_t : t \in \mathbb{N} \cup \{0\}\}$ such that $S_0 = 0$ (this is taken arbitrarily) and

$$S_t = \sum_{j=1}^{t} X_j, \quad t \geq 1,$$

where $\{X_j : j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables, which we shall assume, without any loss of generality, that they are centered and with variance $0 < \sigma^2 < \infty$. So why would we need to give it another

name? What is the difference between this processes and the so-called Brownian motion? To see this, we first see some of the common properties that share the class of random walks with finite variance jumps:

1. The first one, which is a choice rather than some intrinsic property of the process S, is that it starts from the origin.

2. The second one, which is a bit more interesting, is that the displacements of the process are independent and stationary; that is, if $0 \leq s < t \leq s' < t'$, then the random variables $S_t - S_s$ and $S_{t'} - S_{s'}$ are independent and the law of $S_t - S_s$ depends only on $t - s$. Indeed, for the independence of the increments, one has that

$$S_t - S_s = \sum_{j=s+1}^{t} X_j, \quad S_{t'} - S_{s'} = \sum_{j=s'+1}^{t'} X_j.$$

   Since the random variables X_j are mutually independent, we conclude that the increments are independent. As for the second part, the fact that the law of the increment $S_t - S_s$ depends only on $t - s$ means, in our setting, that the law depends only on the number of variables X_j involved. Since they are independent and identically distributed, the claim follows.

3. The last property, but not less important, is that, due to the Central Limit Theorem, for $t \geq 0$ large enough, and roughly speaking,

$$S_t \sim \mathcal{N}(0, \sigma^2 t).$$

   In other words, the long term behaviour of the random variable S_t is described by a centered Gaussian random variable with variance $\sigma^2 t$. Since it depends linearly with time, one can say that the process is diffusive in the long term.

Therefore, it seems that, when the right scales are considered, all random walks behave in the same way (modulo some constant). This is the content of Donsker's Invariance Principle (Theorem 2.2), which we state below. Before doing so, we must first define mathematically what a Brownian motion is.

Definition 2.1. A stochastic process $B = \{B_t : t \in \mathbb{R}_+\}$ is a one-dimensional Brownian motion if:

1. $B_0 = 0$ almost surely.

2. For any $k \in \mathbb{N}$ and any $0 \leq t_1 < \cdots < t_k < \infty$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent.

3. For any $0 \leq s < t < \infty$, the random variable $B_t - B_s$ is normally distributed with zero mean and variance $\sigma^2(t - s)$ for some constant $0 < \sigma < \infty$.

4. The sample paths of the process are continuous everywhere with probability one.

The process B is said to be a standard Brownian motion if $\sigma = 1$.

Observe that many of the properties of the random walk are shared by the Brownian motion. An additional property has been added, which is that the sample paths of the process are continuous with probability one, however, this is not so important, since, if the first three properties are satisfied, one can find a version of the process satisfying the fourth one.

In the case of the random walk, we provided a class of processes (which were determined by the sequence of random variables $\{X_j : j \in \mathbb{N}\}$) that satisfied the first three properties. However, the same cannot be done in the case of the Brownian motion, which can be thought as a continuous time version of the random walk. Hence, we have to first check that such process exists. This is, as well, part of the content of Donsker's Theorem, which we now state.

Theorem 2.2 (Donsker's Invariance Principle)**.** *Let $\{X_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed centered random variables with unitary variance. Then the random (continuous) functions*

$$Y_t^{(n)} = \frac{1}{\sigma\sqrt{n}}\tilde{S}_{nt}, \quad 0 \leq t \leq 1,$$

where

$$\tilde{S}_t = \sum_{j=1}^{[t]} X_j + (t - [t])X_{[t]+1}, \quad \tilde{S}_0 = 0,$$

converge weakly to a standard one-dimensional Brownian motion, where $[t]$ denotes the integer part of t. In other words, if P_n are the laws of the random functions $Y_t^{(n)}$, then there is a probability measure P (the Wiener measure) over the space of real continuous functions on $[0,1]$, $C[0,1]$, fulfilling the properties from Definition 2.1 and such that $P_n(G) \to P(G)$ for any Borel set G of $C[0,1]$ with $P(\partial G) = 0$, being ∂G the boundary of G.

The process $\tilde{S}$, which resembles quite a lot S, is the linear interpolation of the latter and hence, a process with continuous sample paths.

The proof of this result (which is a result of convergence of probability measures), relies, mainly, on Prohorov's Theorem, which gives a characterization of the family of laws induced by the family of random functions $\{Y^{(n)} : n \in \mathbb{N}\}$, with $Y^{(n)} = \{Y_t^{(n)} : t \in [0,1]\}$ in terms of the topological properties of the space $C[0,1]$, and the fact that the finite dimensional distributions of a continuous stochastic process determine its law (we refer to [1, Theorems 5.1, 5.2 and p. 84] for a proof of these claims). A proof of Theorem 2.2 for a particular case of random walk is given in [2], and a general proof can be found in [1, Section 8] as well.

With this, we have given an answer to the first of the two questions and now can address the problem of defining objects like (1).

3. Stochastic differential equations

Before trying to define the concept of solution to equations like (1), which are known as stochastic differential equations (SDEs), we shall first see how one gets to the point of having to consider such objects.

To do so, let us consider an ordinary differential equation (ODE) of the form

$$dX_t = f(t, X_t)\, dt, \quad t \geq 0, \tag{2}$$

modeling some phenomena which we are interested in and where $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is some good enough function.

In some cases, the description given by the ODE might be a bit too simple or might not take into account some factors which might have been neglected due to a simplification or due to the fact that we cannot easily control them. To solve this, one can discretize the ODE and add a random perturbation which might evolve with time, say $V = \{V_t : t \in \mathbb{R}_+\}$, leading to

$$X_{t+\Delta t} - X_t = f(t, X_t)\Delta t + g(t, X_t)\Delta V_t, \quad \Delta V_t = V_{t+\Delta t} - V_t,$$

where $g\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is some function modelling the intensity of the random perturbation. Usually, this introduced noise accounts for the superposition of several small (of finite variance) factors which cannot be controlled. Hence, and due to the Central Limit Theorem, we can assume that the law of the increments ΔV_t is normally distributed with vanishing mean (since the mean trajectories should coincide with the one modeled by (2)) and with variance Δt. The linear dependence on time in the variance is chosen because, in most scenarios, the observed perturbation can be said to be diffusive.

One can assume, as well, that the random perturbations in discrete time, $\Delta V_0, \Delta V_{\Delta t}, \dots$ are uncorrelated or independent since they are supposed to be rapidly varying and hence, what happens in one time interval might not significantly interfere on what happens in some other time interval.

With all this, one concludes that the best choice for the process V is a standard Brownian motion. The only think left to do is to take the limit $\Delta t \to 0$ to obtain, formally speaking,

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t.$$

However, and as mentioned in the introduction, the differential dB_t makes no sense as a classical one. To solve this problem, one writes the SDE in its integral form

$$X_t = X_0 + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s. \tag{3}$$

So the only thing left to do is to give a meaning to expressions $\int_0^t X_s\, dB_s$ (stochastic integral) for a suitable class of stochastic processes $X = \{X_t : t \in \mathbb{R}_+\}$ to solve the problem.

3.1 Stochastic integrals

The first idea to approach such integrals is to use the already developed theory of integration with respect to functions (Lebesgue–Stieltjes integral) to define such integrals pathwise. However, the fact that the sample paths of the Brownian motion are of unbounded variation preclude this option.

For this purpose, a new theory of integration needs to be developed. As in the case of the Riemann–Stieltjes integral, we will be considering sums of the form

$$\sum_{j=0}^{n-1} X_{t_j^*}(B_{t_{j+1}} - B_{t_j}), \tag{4}$$

where $0 = t_0 < \cdots < t_n = T$ is a partition of a finite time interval $[0, T]$ and where $t_j^* \in [t_j, t_{j+1})$, $j = 0, \dots, n-1$. Ideally, one would want the above sums to converge to the same limit (this limit might be in probability or in mean square, for instance) no matter what choice of t_j^* is made. Unfortunately, this is not the case, leading to different definitions of the stochastic integral depending on the choice of the midpoints t_j^*, $j = 0, \dots, n-1$. In this work, we will be considering the left endpoint approximations ($t_j^* = t_j$), which lead to the Itô integral.

As one might expect, this integral will not be defined for any process X. Returning to the discretization of the SDE, we have that the information we have on the process X at time $t+\Delta t$ can be determined by the information we have on X_t and the information we have on the driving noise (in our case, the Brownian motion) at time $t+\Delta t$. At the same time, the information we have of X_t depends on the information one has on $X_{t-\Delta t}$ and so on. All in all, we see that we can infer the information of X_t at time t by knowing the entire information of the driving process B until that time. In particular, the information we have on X_t does not depend on the information we have at time s for $s > t$, so the process X cannot see into the future. In this case, we say that the process X must be adapted to the filtration generated by the driving noise (the information we have on X_t depends on the history of the noise until that time).

Another natural hypothesis on the process X is that it must be integrable in some sense so that we can talk about its integral. More precisely, we will require that

$$||X||^2 := \mathbb{E}\left[\int_0^T X_s^2\, ds\right] < \infty.$$

For this integral to be well defined, we will require, as well, the process X, thought as a map $X\colon \Omega\times[0,\,T] \to \mathbb{R}$, $(\omega,\,t) \mapsto X(\omega,\,t) = X_t(\omega)$, where Ω is the sample space, to be jointly measurable with respect to the corresponding σ-fields.

When all these hypothesis are fulfilled, one can show that integrals like $\int_0^t X_s\, dB_s$ can be defined as an $L^2(\Omega)$-limit (mean square limit) of Riemann–Stieltjes sums. To show this, and as it is customary in this type of constructions, one first defines a class of simple functions of the form

$$\phi(\omega,\,t) = \phi_t(\omega) = \sum_{j=0}^{n-1} e_j(\omega)\mathbb{I}_{[t_j,\,t_{j+1})}(t), \tag{5}$$

where $\{e_j : j = 0,\,\dots,\,n-1\}$ are bounded random variables such that the information we have on e_j depends only on the history of the Brownian motion (the driving noise) until time t_j and $0 = t_0 < \cdots < t_n = T$. For such functions, the integral with respect to the Brownian motion is defined as the sum (4), where $X_{t_j^*}$ must be replaced by e_j.

Next, one checks that $||\cdot||$ defines a norm on the space of processes X satisfying the previously mentioned hypothesis and that such normed space (from now on, the space of Itô integrable processes) is complete.

Finally, one shows that any process X in the normed space can be approximated by simple functions (5), which allows us to define the integral $\int_0^T X_s\, dB_s$ as an $L^2(\Omega)$-limit of integrals of simple processes. To justify this last step, a crucial result for step functions (which also holds for general Itô integrable processes X) is needed. We shall state the result, as it will be useful in the future for other purposes.

Theorem 3.1 (Isometry formula)**.** *For any Itô integrable process X, we have*

$$\mathbb{E}\left[\left(\int_0^T X_s\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T X_s^2\, ds\right].$$

As its name says, the previous result asserts that the stochastic integral with respect to the Brownian motion establishes an isometry between the space of square integrable random variables, $L^2(\Omega)$, and the space of Itô integrable functions. For a detailed construction of the stochastic integral, we refer to Chapter 3 of [3] and [4].

Another important feature of this integral is that, when X is a deterministic Itô integrable process (that is, the map $X : \Omega \times [0, T]$ is constant in the first argument), one has that the process $I = \{I_t : t \in [0, T]\}$ defined by $I_t = \int_0^t X_s \, dB_s$ is a Gaussian process. More precisely,

Theorem 3.2. *If $X = f = \{f_t : t \in [0, T]\}$ is a deterministic Itô integrable process, then I is a centered Gaussian process with independent increments such that, for each $0 \leq s < t \leq T$,*

$$I_t - I_s = \int_s^t f_s \, dB_s \sim \mathcal{N}\left(0, \int_s^t f_u^2 \, du\right).$$

That is, the increment is normally distributed with zero mean and variance $\int_s^t f_u^2 \, du$.

With this, the task of giving a meaning to expressions like (3) has been fulfilled. However, we have not provided any practical way of computing stochastic integrals. This will be the purpose of the Itô formula (see [3, Chapter 3] again or [4, Chapter 4] for a proof of this result), which can be thought as a chain rule or as an integration by parts formula, depending on whether you consider the differential or integral form.

Theorem 3.3 (Itô formula)**.** *Let $X = \{X_t t \in [0, T]\}$ be a process defined by*

$$dX_t = f_t \, dt + g_t \, dB_t$$

or, in integral form,

$$X_t = X_0 + \int_0^t f_s \, ds + \int_0^t g_s \, dB_s,$$

where $f = \{f_t : t \in [0, T]\}$ is a process integrable with respect to the Lebesgue measure with probability one and $g = \{g_t : t \in [0, T]\}$ is an Itô integrable process, and let $F : [0, T] \times \mathbb{R}$, $(t, x) \mapsto F(t, x)$ be a $\mathcal{C}^{1,2}$ function (continuously differentiable with respect to the first argument and twice continuously differentiable with respect to the second one). Then, if $Y_t = F(t, X_t)$,

$$dY_t = \frac{\partial F}{\partial t}(t, X_t) \, dt + \frac{\partial F}{\partial x}(t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, X_t)(dX_t)^2$$

or, in integral form,

$$Y_t = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) \, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) \, dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)(dX_s)^2.$$

In the previous theorem, the differentials dX_t and $(dX_t)^2$ can be treated as if they were finite real quantities by using the rules $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$ and $(dB_t)^2 = dt$. Then, for instance, we have that

$$\int_0^t \frac{\partial F}{\partial x}(s, X_s) \, dX_s = \int_0^t \frac{\partial F}{\partial x}(s, X_s) f_s \, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) g_s \, dB_s,$$

and

$$\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)(dX_s)^2 = \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) g_s^2 \, ds.$$

With this, we can now begin to study stochastic differential equations.

3.2 An existence and uniqueness result

The first thing one must check when one studies equations like (3) (or its differential form), is to make sure that there is at least one solution and, if possible, to see that it is unique. It turns out that, under similar hypothesis to the ones used in Picard's Theorem on the processes g and f, one can show that there is a unique stochastic process X satisfying equation (3). However, the uniqueness is understood in the sense that any other process satisfying the SDE is a modification of our solution.

More particularly, we require f and g to be Lipschitz functions and of linear growth with respect to the second variable for each $t \in [0, T]$:

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq C|x-y|, \quad |f(t,x)| + |g(t,x)| \leq D(1+|x|),$$

for some positive constants C and D, and the initial condition X_0 to be deterministic (this last hypothesis can be relaxed by considering any square integrable initial condition satisfying some measurability properties). For a precise statement of the result and a proof, we refer to [4, Theorem 5.2.1].

With all this, we can finally study equations like the Langevin one, equation (1), when the noise $\dot{F}_t$ is identified with the differential of the Brownian motion. In the following section we treat a particular case of such equations and compute some observable quantities.

3.3 The case of the Langevin equation

Let us consider equation (1) when $\dot{F}_t\, dt = \sigma\, dB_t$ for some real constant σ. That is, we consider the equation

$$d\dot{X}_t = -\mu \dot{X}_t\, dt + \sigma\, dB_t,$$

for some positive constant μ and some real constant σ. The theorem of existence and uniqueness of solutions tells us that, for each $T \geq 0$ and any deterministic initial condition $\dot{X}_0$, there is a unique process (modulo modifications) $\dot{X} = \{\dot{X}_t : t \in [0, T]\}$ satisfying the above equation. To give an explicit formula for $\dot{X}_t$, we multiply the SDE by the integrating factor $e^{\mu t}$, which leads to

$$e^{\mu t}\, d\dot{X}_t + \mu e^{\mu t} \dot{X}_t\, dt = e^{\mu t} \sigma\, dB_t.$$

The usual product rule would tell us that the left-hand side can be identified with $d(e^{\mu t}\dot{X}_t)$. However, this might not be true in the context of stochastic processes. To make sure that this holds, we apply Itô's formula to the function $F(t,x) = xe^{\mu t}$, for which we have

$$\frac{\partial F}{\partial t}(t,x) = \mu x e^{\mu t}, \quad \frac{\partial F}{\partial x}(t,x) = e^{\mu t}, \quad \frac{\partial^2 F}{\partial x^2}(t,x) = 0.$$

So, indeed, we have that

$$d(e^{\mu t}\dot{X}_t) = e^{\mu t}\, d\dot{X}_t + \mu e^{\mu t}\dot{X}_t\, dt = e^{\mu t}\sigma\, dB_t$$

or, in integral form,

$$\dot{X}_t e^{\mu t} = \dot{X}_0 + \sigma \int_0^t e^{\mu s}\, dB_s.$$

Which simplifies to

$$\dot{X}_t = \dot{X}_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)}\, dB_s.$$

With this and other results like the isometry formula, we can compute some observable quantities like the mean, the variance and the covariance. A straightforward computation using the isometry formula shows that

$$\mathbb{E}[\dot{X}_t] = \dot{X}_0 e^{-\mu t}, \quad \mathbb{E}[\dot{X}_t^2] = \dot{X}_0^2 e^{-2\mu t} + \frac{\sigma^2}{2\mu}(1 - e^{-2\mu t}), \quad \mathrm{Var}(\dot{X}_t) = \frac{\sigma^2}{2\mu}(1 - e^{-2\mu t}). \tag{6}$$

Finally, for $0 \leq s < t$, we have, by letting $I_t = \int_0^t e^{\mu u}\, dB_u$ (observe that the stochastic integral involved is the one of a deterministic function, so we are under the hypothesis of Theorem 3.2),

$$\begin{aligned}
\mathrm{Cov}(\dot{X}_t, \dot{X}_s) &= \mathbb{E}[(\dot{X}_t - \mathbb{E}[\dot{X}_t])(\dot{X}_s - \mathbb{E}[\dot{X}_s])] \\
&= \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_t I_s] \\
&= \sigma^2 e^{-\mu(t+s)} \mathbb{E}[(I_t - I_s) I_s] + \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_s^2] \\
&= \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_t - I_s] \mathbb{E}[I_s] + \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_s^2] \\
&= \sigma^2 e^{-\mu(t+s)} \int_0^s e^{2\mu u}\, du \\
&= \frac{\sigma^2}{2\mu}(e^{\mu(s-t)} - e^{-\mu(t+s)}),
\end{aligned}$$

where we have used Theorem 3.2 and the isometry formula. Hence, for any $s, t \in [0, T]$,

$$\mathrm{Cov}(\dot{X}_t, \dot{X}_s) = \frac{\sigma^2}{2\mu}(e^{-\mu|t-s|} - e^{-\mu(t+s)}). \tag{7}$$

Moreover, Theorem 3.2 tells us that the process $\dot{X}$ is Gaussian with mean and covariance functions given by the first term in (6) and (7), respectively, and that, for each $t \in [0, T]$, $\dot{X}_t$ is a normal random variable with mean and variance given by the first and last terms in (6).

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Extensions of the Calderón–Zygmund theory

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Resum *(CAT)*
Les eines desenvolupades en la dècada de 1950 per Calderón i Zygmund ens permeten demostrar que algunes integrals singulars estan ben definides i fitades en els espais L^p. Tot i que l'espai euclidià fos el context original on totes aquestes idees es varen desenvolupar, aquestes propietats es generalitzen a altres espais mètrics de mesura i a integrals singulars de valors vectorials. Al llarg de les dècades, la teoria ha anat guanyant en abstracció i interès. Encara avui en dia, hi ha operadors que s'escapen de l'abast de la teoria, com és l'operador diàdic esfèric maximal.

Abstract *(ENG)*
The tools developed in the 1950s by Calderón and Zygmund enable us to prove that certain singular integrals are well defined and bounded in L^p spaces. Although the Euclidean space was the original context where all these ideas were developed, these properties generalise to other measure metric spaces and to vector-valued singular integrals. Along the decades, the theory has been acquiring abstraction and luring attention. Even nowadays, there are operators that fall outside the scope of the theory, for instance the dyadic spherical maximal operator.

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1. Introduction

By "singular integral operators" we mean, in the first instance, convolution operators in $\mathbb{R}^n$ the kernel function of which presents a singularity, say, at the origin. Namely, we think of operators of the kind

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy, \quad x \in \mathbb{R}^n,$$

for some given function K that blows up at the origin. Singular integrals show up in a number of problems of analytic nature. For instance, they generate solutions of some partial differential equations, they arise in complex analysis, they underpin apparently unrelated settings in geometric measure theory, etc. See Figure 1 for an illustrative example.

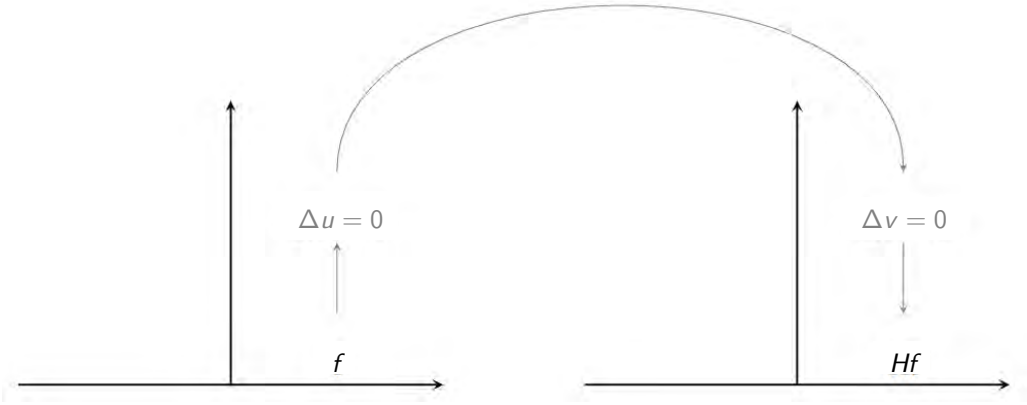

Figure 1: Appearance of the Hilbert transform (the most iconic example of singular integral in $\mathbb{R}$) in Dirichlet's problem for the Laplace equation. First, let f be defined on the axis $y = 0$. Obtain u such that $\Delta u = 0$ in the upper half plane and f is the boundary value of u. Then, get the conjugate harmonic function v of u (the one that turns $u(x,y) + i\,v(x,y)$ into a holomorphic function on the complex plane). Finally, obtain the Hilbert transform of f, Hf, by computing the limit $\lim_{y\to 0} v(x,y)$.

For decades, analysts felt uncomfortable when utilising singular integrals because there was no knowledge regarding their boundedness properties. Were they handling continuous operators on L^p spaces or not? In order to answer this question, Harmonic Analysis is the natural framework.

In the middle and end of the 20^{th} century, the field experienced a burst. Brilliant mathematicians contributed to the expansion of the theory concerning singular integrals. Calderón, Zygmund, Bourgain and Stein are just some of the most influential driving forces in the field, who built upon the work of other great figures like Hardy, Littlewood and Paley.

In the literature, singular integrals are ubiquitous, as they serve to step forward at stages within problems of different natures. Despite this, theory of singular integrals is often just partially explained and treated as an instrument. In this document, we centre them in the spotlight.

2. Calderón–Zygmund theory

The Calderón–Zygmund theory was developed originally in the setting of $\mathbb{R}^n$ in the 1950s, set off by the collaborative breakthrough paper [3] published in 1952. It aimed to prove boundedness of singular convolution-type operators on spaces of functions (mainly L^p spaces) built over $\mathbb{R}^n$.

The starting point is a decomposition lemma that, given an integrable function, enables to split the domain $\mathbb{R}^n$ into a set where the function is bounded, and another set where, although the function may be unbounded, it is controlled in average.

Lemma 2.1 (Calderón–Zygmund lemma in $\mathbb{R}^n$; see [5, Chapter 1, Theorem 4]). *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. There exists a partition $\mathbb{R}^n = F \sqcup \Omega$, such that*

(a) $|f(x)| \leq \lambda$ a.e. $x \in F$, and

(b) Ω can be written as a countable union of cubes Q_k with disjoint interior $\Omega = \bigsqcup_{k\in\mathbb{N}} Q_k$, moreover satisfying

$$\lambda \leq \frac{1}{|Q_k|} \int_{Q_k} |f(x)|\, dx \leq 2^n \lambda, \quad \forall\, k \in \mathbb{N}. \tag{1}$$

Proof. Mesh $\mathbb{R}^n$ into a set of cubes $\{Q_k^0\}_{k\in\mathbb{N}}$ with disjoint interiors and of the same size, large enough so that the averages of $|f|$ are bounded above by the given λ on all of the cubes in the mesh:

$$\frac{1}{|Q_k^0|} \int_{Q_k^0} |f(x)|\, dx < \lambda, \quad \forall\, k \in \mathbb{N}.$$

This is possible because f is integrable,

$$\frac{1}{|Q_k^0|} \int_{Q_k^0} |f(x)|\, dx \leq \frac{\|f\|_1}{|Q_k^0|},$$

so choose the size of the cubes such that $|Q_k^0| > \frac{\|f\|_1}{\lambda}$.

We are going to run an algorithm in order to construct F and Ω. Set $\Omega = \emptyset$ and the step $s = 1$. We split each of the cubes Q_k^0 into 2^n dyadic descendant cubes of the same size Q_k^1.

Case 1: For each descendant cube in step s (that is, for each $k \in \mathbb{Z}$), if

$$\frac{1}{|Q_k^s|} \int_{Q_k^s} |f(x)|\, dx > \lambda,$$

then Q_k^s is selected to take part in the set Ω, so update $\Omega^{\text{new}} = \Omega^{\text{old}} \cup Q_k^s$. For such a cube Q_k^s, assume that Q_r^{s-1} is its direct ancestor. Then, by (2) and the fact that Q_r^{s-1} fell into Case 2,

$$\lambda < \frac{1}{|Q_k^s|} \int_{Q_k^s} |f(x)|\, dx \leq \frac{2^n}{|Q_r^{s-1}|} \int_{Q_r^{s-1}} |f(x)|\, dx \leq 2^n \lambda, \tag{2}$$

which proves (1) for Q_k^s.

Case 2: Instead, if

$$\frac{1}{|Q_k^s|} \int_{Q_k^s} |f(x)|\, dx \leq \lambda,$$

then we iterate and further divide Q_k^s into 2^n identical descendant cubes (each with half the sidelength of the ancestor), and check into which of the two cases each of them falls.

Update $s^{\text{new}} = s^{\text{old}} + 1$ and let the algorithm run recursively. This way, we obtain the desired partition $\mathbb{R}^n = F \sqcup \Omega$, Ω being the union of all those cubes that fell into Case 1, and F being the complement of Ω. Plus, (b) has been verified for all cubes Q_k^s that were selected for Case 1. Fact (a) follows from the Lebesgue differentiation theorem: if $x \in F$, this means that there exists a sequence of nested dyadic cubes containing x, $(Q^s_{k(s)})_{s\in\mathbb{N}}$, $(Q^s_{k(s)}) \supset (Q^{s+1}_{k(s+1)})$ being direct dyadic descendants $\forall\, s \in \mathbb{N}$, such that all of these cubes fell into Case 2, implying that

$$f(x) = \lim_{s\to\infty} \frac{1}{|Q^s_{k(s)}|} \int_{Q^s_{k(s)}} |f(y)|\, dy \leq \lambda. \qquad \square$$

This decomposition of the domain $\mathbb{R}^n$ of f leads to a useful decomposition of the function f itself. By defining

$$g(x) := \begin{cases} f(x), & x \in F, \\ \dfrac{1}{|Q_k|} \displaystyle\int_{Q_k} f(x)\, dx, & x \in Q_k, \end{cases}$$

and $b(x) := f(x) - g(x)$, we reach the following corollary.

Corollary 2.2 (See [5, Chapter 2, Theorem 1]). *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. There exists a decomposition of f as sum of two functions, $f = g + b$ such that:*

(a) $g(x) \leq 2^n\lambda$ *a.e.* $x \in \mathbb{R}^n$,

(b) $\dfrac{1}{|Q_k|} \displaystyle\int_{Q_k} b(x)\, dx = 0 \ \forall\, k \in \mathbb{N}$,

(c) $\dfrac{1}{|Q_k|} \displaystyle\int_{Q_k} |b(x)|\, dx \leq 2^n\lambda \ \forall\, k \in \mathbb{N}$,

(d) $\operatorname{supp}(b) = \bigsqcup_{k\in\mathbb{N}} Q_k$ *and*

(e) $b \leq f$ *a.e.*

The functions g and b are usually referred to as the "good" and the "bad" part of f. Corollary 2.2 is the key ingredient to prove Theorem 2.4, that allows us to bound singular integral operators. However, as one may guess, we first need to make some assumption on the regularity of the singular kernel function. The minimal known hypothesis that succeeds is the so-called Hörmander's condition.

Definition 2.3. A convolution kernel K on $\mathbb{R}^n$ is said to satisfy *Hörmander's condition* if

$$B := \sup_{|y|>0} \int_{|x|\geq 2|y|} |K(x-y) - K(x)|\, dx < \infty. \tag{3}$$

Since the integral is computed over the region $\{x \in \mathbb{R}^n : |x| > 2|y|\}$, the singularity of the kernel is avoided both for $x - y$ and for x. In some sense, we are asking that the global variation of the kernel is not so wild that is not integrable. Nevertheless, Hörmander's condition is usually seen as a weakened version of the stronger condition

$$|\nabla K(x)| \leq \frac{C}{|x|^{n+1}},$$

for all $x \in \mathbb{R}^n$ away from the origin. All in all, here is the theorem that gives meaning to the theory. In the literature, one can find many variations and consequences of it.

Theorem 2.4 (See [5, Chapter 2, Sections 2 and 3]). *Let T be a linear operator such that there exists a measurable kernel function K such that*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

converges absolutely whenever $f \in L^2(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$. Suppose the following:

(i) T is bounded on $L^2(\mathbb{R}^n)$: there exists $A > 0$ such that for all $f \in L^2(\mathbb{R}^n)$, $\|Tf\|_2 \leq A\|f\|_2$.

(ii) The kernel K satisfies Hörmander's condition (3) with constant B.

Then,

(a) T is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and

$$\|Tf\|_p \leq C_{n,p}\|f\|_p,$$

for $f \in L^p(\mathbb{R}^n)$ and $C_{n,p} > 0$ only depending on n, p, A and B.

(b) T is weak-type $(1,1)$, i.e., for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$,

$$\lambda|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C_n\|f\|_1,$$

where $C_n > 0$ is a constant only depending on the dimension n, A and B.

The strategy for the proof is, accounting for the boundedness assumption on the Hilbert space $L^2(\mathbb{R}^n)$, using the Calderón–Zygmund lemma to first show (b), i.e., that T is weak-type $(1,1)$. After that, one can use the Marcinkiewicz interpolation theorem between $p = 1$ and $p = 2$ to get (a) for $1 < p \leq 2$. Eventually, a duality argument covers the dual range $2 \leq p < \infty$.

3. Extensions of the theory

In view of Theorem 2.4, it is natural to wonder if it admits generalisations to other settings. Indeed, under suitable conditions, it is possible to extend the theorem, on the one hand, to other measure metric spaces, and on the other hand, to vector-valued functions. The first setting is useful, for example, in the theory of parabolic PDEs, whereas the latter generalisation turns out to be handy to study maximal operators or operators of the kind "square functions". In this section, we present such an abstraction accounting for the combination of both extensions.

Definition 3.1. A measure metric space $((X,d),\Sigma,\mu)$ is said to have the *doubling property* if

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x)), \quad \forall\, r > 0,\, x \in X,$$

$C > 0$ being a universal constant for the space X. This is, measures of dilated balls are comparable.

The doubling property is crucial if we need available inequalities of the kind (2). Along this section, $((X,d),\Sigma,\mu)$ denotes a generic σ-finite measure space over a metric space equipped with a regular measure enjoying the doubling property.

Next, note that in an arbitrary metric space, cubes are not available anymore, but only balls. Therefore, the proof of Lemma 2.1 completely breaks apart, since it relies heavily on meshing $\mathbb{R}^n$ into cubes. This implies that the strategy to get a lemma of the same flavour has to be totally different. To this end, the Hardy–Littlewood maximal function aids.

Definition 3.2. Let $((X,d),\Sigma,\mu)$ be a measure metric space and let $f\in L^1_{\mathrm{loc}}(X)$ be a locally integrable function. The *centred Hardy–Littlewood maximal function* of f is defined as

$$\mathfrak{M}f(x):=\sup_{r>0}\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)|\,d\mu(y). \tag{4}$$

Similarly, the *uncentred Hardy–Littlewood maximal function* of f reads as

$$\mathfrak{M}^{\mathrm{unc}}f(x):=\sup_{B\ni x}\frac{1}{\mu(B)}\int_{B}|f(y)|\,d\mu(y),$$

where the supremum is taken over all balls B containing x.

When the doubling property is in force, then the centred and uncentred version are easily checked to be comparable. It is also remarkable to note that the Hardy–Littlewood maximal function defines a bounded operator on L^p spaces, $1<p<\infty$ ([5, Chapter 1, Theorem 1]). In fact, in order to show L^p-boundedness for a broad class of so-called Calderón–Zygmund operators (those under the hypotheses of Theorem 2.4 or Theorem 3.5), one can first show, as pointed out, that the Hardy–Littlewood maximal function is L^p-bounded, and then use this specific result to prove L^p-boundedness for the broad class of Calderón–Zygmund operators.

Lemma 3.3 (Calderón–Zygmund lemma in the general setting; see [7, Chapter 1, Theorem 2])**.** *Let $f\in L^1(X)$ and $\lambda>0$. There exists a partition of the space $X=F\sqcup\Omega$, F being a closed set and Ω an open set, such that*

(a) $|f(x)|\le\lambda$ a.e. $x\in F$, and

(b) Ω can be written as a countable disjoint union of smaller sets $\Omega=\bigsqcup_{k\in\mathbb{N}}\Omega_k$ moreover satisfying

$$\frac{1}{\mu(\Omega_k)}\int_{\Omega_k}|f(x)|\,d\mu(x)\le C\lambda,\quad\forall\,k\in\mathbb{N},$$

for some constant $C>0$.

Proof. Let $f\in L^1(X)$ and fix $\lambda>0$. Choose $F:=\{x\in X:\mathfrak{M}f(x)\le\lambda\}$ and so $\Omega:=\{x\in X:\mathfrak{M}f(x)>\lambda\}$, being respectively closed and open, because $\mathfrak{M}f(x)$ is a continuous function of x.

By the Lebesgue differentiation theorem, for a.e. $x\in F$,

$$\lambda\ge\mathfrak{M}f(x)=\sup_{r>0}\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)|\,d\mu(y)\ge\lim_{r\to0}\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)|\,d\mu(y)=|f(x)|,$$

so (a) is shown.

Let us introduce some notation. For a ball $B=B_r(x)$ centred at x with radius r and for some universal constants $0<C^*<C^{**}$, denote by $B^*:=B_{C^*r}(x)$ and $B^{**}:=B_{C^{**}r}(x)$ the centred dilations by factors C^* and C^{**}, respectively. In order to prove (b), we use a Vitali-type covering lemma ([7, Chapter 1, Lemma 2]): given the closed set F, there exists a sequence of balls $(B_k)_{k\in\mathbb{N}}$ and two families of each dilations (or universal dilation constants $0<C^*<C^{**}$), $(B_k^*)_{k\in\mathbb{N}}$ and $(B_k^{**})_{k\in\mathbb{N}}$, such that

(a) $(B_k)_{k\in\mathbb{N}}$ are pairwise disjoint,

(b) $\bigcup_k B_k^*=F^c$, and

(c) $B_k^{**}\cap F\neq\emptyset$, $\forall\,k$.

It is convenient to extract another sequence of sets. Take the first element in $(B_k^*)_{k\in\mathbb{N}}$ and define $Q_1 := B_1^*$. Next, define $Q_2 := B_2^* \setminus (Q_1)$. By an inductive process, build

$$Q_k := B_k^* \setminus \left(\bigcup_{j=1}^{k-1} Q_j \right).$$

It is directly deduced that the sets Q_k satisfy $\bigcup_k Q_k = F^c$ just like the B_k^*, although with the advantage that the Q_k are pairwise disjoint. The downside, compared to the B_k^*, is that the Q_k are no longer balls, but other less elementary sets. The name Q_k of such new sets is inspired by their role in the proof of Theorem 3.5, which mimics the one carried out by the cubes in the proof of the $X = \mathbb{R}^n$ case.

Now, for each B_k in the sequence $(B_k)_{k\in\mathbb{N}}$, choose a point $p_k \in B_k^{**} \cap F$. By the definition of F,

$$\begin{aligned} \lambda \geq \mathfrak{M}f(p_k) \geq C^{\text{unc}} \mathfrak{M}^{\text{unc}} f(p_k) \geq \frac{C^{\text{unc}}}{\mu(B_k^{**})} \int_{B_k^{**}} |f(x)|\, d\mu(x) \\ \geq \frac{C^{\text{unc}}}{\mu(B_k^{**})} \int_{Q_k} |f(x)|\, d\mu(x) \geq \frac{C^{\text{unc}}}{C^{\text{dp}}} \frac{1}{\mu(Q_k)} \int_{Q_k} |f(x)|\, d\mu(x), \end{aligned} \tag{5}$$

where C^{dp} is the constant from the doubling property (see Definition 3.1) and C^{unc} is the constant in the equivalence

$$\mathfrak{M}f \leq \mathfrak{M}^{\text{unc}} f \leq C^{\text{unc}}\, \mathfrak{M}f.$$

In fact, $C^{\text{unc}} = (C^{\text{dp}})^{-1}$. The two last inequalities in (5) stem from the fact that $B_k \subseteq Q_k \subseteq B_k^{**}$ and the doubling property: $\mu(Q_k) \leq \mu(B_k^{**}) \leq C^{dp}\mu(B_k) \leq C^{\text{dp}}\mu(Q_k)$. Since $(Q_k)_{k\in\mathbb{N}}$ partition Ω, $\Omega = \bigsqcup_k \Omega_k \equiv \bigsqcup_k Q_k$, the proof is complete. □

Note that this proof unveils the precise identity of the sets F and Ω, which are defined in terms of the Hardy–Littlewood maximal function.

In exactly the same way as in Corollary 2.2, the Calderón–Zygmund decomposition of an integrable function $f \in L^1(X)$ as $f = g + b$ is deduced.

We mentioned that we wish our generalised theorem to hold for vector-valued functions. The construction of the L^p spaces for such functions is nowadays standard ([4, Chapter 5]). Let us denote by $L^p_B(X)$ the Lebesgue space of L^p-integrable functions on some measure space X and taking values in the Banach space B. This is, for $1 \leq p < \infty$, set

$$L^p_B(X) := \left\{ F\colon X \to B : \int_X \|F(x)\|_B^p \, d\mu(x) < \infty \right\},$$

whereas for $p = \infty$,

$$L^\infty_B(X) := \left\{ F\colon X \to B : \operatorname*{ess\,sup}_{x\in X} \|F(x)\|_B < \infty \right\}.$$

Additionally, denote by $\mathcal{L}(A, B)$ the Banach space of all linear and continuous maps between Banach spaces A and B.

Note that what has been presented so far in this section also applies to Banach-valued functions.

In order not to scatter away from the theory, we need to upgrade Hörmander's condition on kernel functions as follows. In particular, note that the kernel is no longer a function, but rather a linear operator between Banach spaces.

Definition 3.4. Let A and B be Banach spaces. An operator kernel K on the product measure space $((X,d),\Sigma,\mu)\times((X,d),\Sigma,\mu)$ taking values in $\mathcal{L}(A,B)$ is said to satisfy *Hörmander's condition* if

$$D := \sup_{y,y_0\in X}\int_{d(x,y)\geq Cd(y,y_0)} \|K(x,y)-K(x,y_0)\|_{\mathcal{L}(A,B)}\,d\mu(x)<\infty, \tag{6}$$

for some constant $C>1$.

Another important remark is that now, the kernel operator involves two entries instead of just one, compared to the convolution operators. The reason for this is that "$x-y$" does not make sense in general measure metric spaces, since they lack the vector space structure. Thus, we get around this issue by inputting two variables $x\in X$ and $y\in X$, with the understanding that the kernel is singular around $x=y$.

Astonishingly, the natural generalisation of Theorem 2.4 turns out to work in this setting as well!

Theorem 3.5 (See [7, Chapter 1, Theorem 3] and [4, Chapter 5, Theorem 3.4])**.** *Let $((X,d),\Sigma,\mu)$ be a measure metric space with the doubling property. Let A, B be Banach spaces and let T be a linear operator which is represented by*

$$TF(x)=\int_X K(x,y)F(y)\,d\mu(y),$$

whenever $F\in L^\infty_A(X)$ with compact support and $x\notin\operatorname{supp}(F)$, where the vector-valued kernel $K\in\mathcal{L}(A,B)$ is measurable in $X\times X$ and locally integrable away from the diagonal. Assume that

(i) T is bounded from $L^q_A(X)$ to $L^q_B(X)$ for a fixed $1<q\leq\infty$: there exists $C_q>0$ such that for all $F\in L^q_A(X)$, $\|TF\|_{L^q_B(X)}\leq C_q\|F\|_{L^q_A(X)}$, and

(ii) the operator kernel K satisfies Hörmander's condition in (6) with constants C and D.

Then,

(a) the operator T has a bounded extension mapping $L^p_A(X)$ to $L^p_B(X)$, with $1<p<q$. Furthermore,

$$\|TF\|_{L^p_B(X)}\leq C_p\|F\|_{L^p_A(X)},\quad 1<p<q,$$

for $F\in L^p_A(X)$ and $C_p>0$ only depending on p, q, C_q, C and D.

(b) The operator T has a bounded weak-type $(1,1)$ extension that satisfies

$$\lambda\,\mu(\{x\in X:\|TF(x)\|_B>\lambda\})\leq C_1\|F\|_{L^1_A(X)},\quad \forall\lambda>0,$$

for $F\in L^1_A(X)$ and $C_1>0$ only depending on q, C_q, C and D.

The proof follows the strategy of that of Theorem 2.4, just this time using Lemma 3.3 instead of Lemma 2.1, and caring about the technical details of working in the general case.

Here is an example of operator that falls under the scope of the theory.

Example 3.6 (Smooth Littlewood–Paley square function)**.** Let P_j be smooth Littlewood–Paley projectors. Namely, the P_j are defined as multipliers on the Fourier side:

$$\widehat{P_jf}(\xi):=\psi(2^{-j}\xi)\hat f(\xi),\quad \forall j\in\mathbb{Z}.$$

Here, ψ is a smooth compactly supported function, the dyadic dilations of which form a partition of unity in frequency. This way, $P_j f$ captures the "part" of f with frequencies around 2^j.

The operator

$$Sf(x) := \left(\sum_{j \in \mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}}$$

is named smooth Littlewood–Paley square function.

First of all, we like to think of the square function as the norm of an operator acting on vector-valued functions $S \colon L^p(\mathbb{R}^n) \to L^p_{\ell^2}(\mathbb{R}^n)$: Define

$$\begin{aligned} P(f) &:= (P_j f)_{j \in \mathbb{Z}} = (\dots, P_{-1} f, P_0 f, P_1 f, \dots) \\ &= (\dots, 2^{-n} \check{\psi}(2^{-1}\xi) * f(x), 2^0 \check{\psi}(2^0 \xi) * f(x), 2^n \check{\psi}(2^1 \xi) * f(x), \dots), \end{aligned}$$

which is a *linear* operator mapping functions to sequences of functions.[1] Accordingly,

$$Sf(x) = \|Pf(x)\|_{\ell^2(\mathbb{Z})}.$$

We brought the square function to the vector-valued setting. At this point, one would attempt to apply Theorem 3.5 to Sf. Nonetheless, a direct application fails to show that Sf is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. It is necessary to combine Theorem 3.5 with a probabilistic trick involving Rademacher random variables to eventually show that Sf is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

4. Beyond the paradigm

Together with the development of the Calderón–Zygmund theory as well as its extensions, new problems arose in the field. In particular, interest was shown in singular measure operators. The reason for this interest relies on the thirst for understanding other appealing problems like the Kakeya problem, the Bochner–Riesz conjecture or the Fourier restriction problem, which still remain mysterious and open. Let us give an example in this direction of an operator that is still not completely understood.

Definition 4.1. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function in $\mathbb{R}^n$. Define the *dyadic spherical maximal function* as

$$\tilde{\mathcal{S}} f(x) := \sup_{k \in \mathbb{Z}} \int_{\mathbb{S}^{n-1}} f(x - 2^k \omega) \, d\sigma(\omega), \quad \forall x \in \mathbb{R}^n, \tag{7}$$

where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere, σ is the surface measure of $\mathbb{S}^{n-1}$ and $\omega \in \mathbb{S}^{n-1}$ is a unit vector.

The (non-maximal) spherical means appear in the expression for the solution to the Cauchy problem of the wave equation in odd space dimension. The interest in studying its maximal versions relies on the availability of a standard strategy to prove pointwise convergence results of the solution to the wave equation towards the initial datum.

The operator (7) is similar to the Hardy–Littlewood maximal function (4) in the sense that, instead of averaging over balls, it averages over spheres. However, the surface measure of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is a singular

[1] One can play the same trick with maximal functions. For instance, $\mathfrak{M} f(x) = \|A(x, \cdot) f\|_{L^\infty(\mathbb{R}_{>0})}$, where $A(x, r) f$ denotes the average of f on the ball centred at x of radius r.

measure, in the sense that all of its mass is concentrated on a null n-Lebesgue measure manifold. Furthermore, (7) can be seen as a convolution of a function f against a (singular) measure, but not another function anymore. This brings obstacles to our understanding of the spherical maximal function, because the Calderón–Zygmund theory from previous sections does not apply anymore.

In this case, the radii are discretised. It is of course of interest to take supremum over the continuum $r > 0$. In that case, the spherical maximal function has been understood deeply and it turns out that the boundedness properties depend on the dimension [1, 6]. Up to the date, we know that this dyadic version defines indeed a bounded operator on $L^p(\mathbb{R}^n)$. Nonetheless, we do not know whether it is weak-type $(1,1)$.

Theorem 4.2 (See [2]). *The dyadic spherical maximal operator $\tilde{\mathcal{S}}f$ is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. This is, for $f \in L^p(\mathbb{R}^n)$,*

$$\|\tilde{\mathcal{S}}f\|_p \leq C_p \|f\|_p,$$

for some constant $C_p > 0$ depending on p and n.

Conjecture 4.3. *The dyadic spherical maximal operator $\tilde{\mathcal{S}}f$ is weak-type $(1,1)$. So for any $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$,*

$$\lambda |\{x \in \mathbb{R}^n : \tilde{\mathcal{S}}f(x) > \lambda\}| \leq C_1 \|f\|_1,$$

for some constant $C_1 > 0$ depending on n.

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Survey on optimal isosystolic inequalities on the real projective plane

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Resum *(CAT)*

Es revisen totes les desigualtats isosistòliques òptimes conegudes al pla projectiu real $\mathbb{RP}^2$, comparant-les amb el cas del 2-tor $\mathbb{T}^2$. Primer s'introdueixen nocions bàsiques de mètriques de Finsler. Després s'enuncien totes les desigualtats isosistòliques conegudes pel cas reversible i se'n dona la idea de prova. Finalment es tracten les desigualtats òptimes pel cas no-reversible. Actualment es coneixen totes les desigualtats òptimes per $\mathbb{T}^2$, tot i que no és així per $\mathbb{RP}^2$. S'hi presenten alguns petits progressos i arguments a favor de la desigualtat conjecturada en el cas encara obert.

Abstract *(ENG)*

All known optimal isosystolic inequalities on the real projective plane $\mathbb{RP}^2$ are surveyed, comparing them to the case of the 2-torus $\mathbb{T}^2$. First, basic notions on Finsler metrics are introduced. Then, all previously known isosystolic inequalities are stated and a sketch of proof is given in the reversible case. Finally, optimal inequalities in the non-reversible case are discussed. All optimal inequalities are currently known for $\mathbb{T}^2$, although this is not the case for $\mathbb{RP}^2$. Some recent minor advances for $\mathbb{RP}^2$ are presented, and some arguments are given in favour of the conjectured inequality in the remaining open case.

Keywords: *real projective plane, systole, isosystolic inequality, Riemannian metric, Finsler metric, Busemann–Hausdorff area, Holmes–Thompson area.*
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1. Riemannian and Finsler metrics

Riemannian manifolds, introduced in the second half of the 19th century by Bernhard Riemann, are manifolds endowed with a scalar product on each tangent space. Usually, one works with an n-dimensional smooth manifold M and a Riemannian metric $g_x\colon T_xM \times T_xM \to \mathbb{R}$, denoting a scalar product that varies smoothly with $x \in M$. This scalar product gives rise to a norm on tangent vectors, by setting $\|v\|_x^g = \sqrt{g_x(v,v)}$, and to a length for curves $\gamma\colon [0,1] \to M$, by setting $\ell_g(\gamma) = \int_0^1 \|\gamma'(t)\|^g_{\gamma(t)}\,\mathrm{d}t$. The scalar product can alternatively be represented in a local chart by a collection of $n \times n$ positive definite and symmetric matrices $(g_{ij}(x))_{ij}$. That way, the canonical Riemannian measure $\mathrm{d}v_g$ of (M,g) in this local chart is given by the formula $\mathrm{d}v_g(x) = \sqrt{\det(g_{ij}(x))}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Finsler manifolds are a generalisation of Riemannian manifolds, where each tangent space is endowed with a norm instead of with a scalar product. These metric structures were first considered in 1918 by Paul Finsler, although the term *Finsler manifold* was coined later by Élie Cartan, in 1934. Usually a norm $\|\cdot\|$ is a map from a vector space to $\mathbb{R}^+ = [0,\infty)$ that fulfils the following conditions: $\|v\| = 0$ only if $v = 0$, $\|\lambda v\| = |\lambda|\|v\|$ for $\lambda \in \mathbb{R}$ and $\|v+v'\| \leq \|v\| + \|v'\|$. In Finsler geometry, non-necessarily symmetric norms are considered more generally by replacing the second property by the condition $\|\lambda v\| = \lambda\|v\|$ for $\lambda \in \mathbb{R}^+$. The structure associated to a varying norm on each tangent space is called a Finsler metric and the norm at some point x is usually denoted by F_x. In analogy to the Riemannian case, one defines the length of a curve $\gamma\colon [0,1] \to M$ by $\ell_F(\gamma) = \int_0^1 F_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t$. However, and in contrast to the Riemannian case, there is no unambiguously defined volume notion for Finsler metrics. Two of the most used ones are the Holmes–Thompson and the Busemann–Hausdorff volumes. The former is related to the standard symplectic form on T^*M, and, the latter, to the Hausdorff measure of a metric space in the symmetric case. From now on, only 2-dimensional manifolds will be considered. Fixing an auxiliary Riemannian metric g on M, the Holmes–Thompson and Busemann–Hausdorff areas are defined as

$$\begin{aligned} \operatorname{area}_{\mathrm{HT}}(M,F) &\coloneqq \frac{1}{\pi}\int_M |B_x^\circ|_g\,\mathrm{d}v_g, \text{ and} \\ \operatorname{area}_{\mathrm{BH}}(M,F) &\coloneqq \pi \int_M \frac{1}{|B_x|_g}\,\mathrm{d}v_g, \text{ respectively.} \end{aligned} \tag{1}$$

Here, $|B_x|_g$ denotes the Riemannian measure of the unit ball $B_x = \{v \in T_xM \mid F_x(v) \leq 1\}$, and B_x° its polar convex body with respect to g_x. Note that a Finsler metric F is uniquely defined specifying the unit spheres $U_x = \{v \in T_xM \mid F_x(v) = 1\}$ at each point $x \in M$.

Definition 1.1. A Finsler metric F on M is said to be reversible if $F_x(v) = F_x(-v)$ for all $(x,v) \in TM$. In other words, F is said to be reversible if all the unit balls are centrally symmetric.

Since a scalar product induces a symmetric norm on each tangent space, Riemannian metrics are a particular case of Finsler metrics. As sketched in [5, Proposition 3.5], the definitions in (1) are independent of the chosen auxiliary Riemannian g, and an easy consequence of the Blaschke–Santaló inequality is the following.

Proposition 1.2. *If F is a reversible Finsler metric on a manifold M, then* $\operatorname{area}_{\mathrm{BH}}(M,F) \geq \operatorname{area}_{\mathrm{HT}}(M,F)$ *and equality holds if and only if F comes from a Riemannian metric.*

2. Isosystolic inequalities

In either the Riemannian or Finsler case, there is a notion of length of curves, and for closed manifolds that are not simply connected one can define the following notion of systole.

Definition 2.1. The systole of a Finsler closed manifold (M, F) which is not simply connected is defined by

$$\operatorname{sys}(M, F) := \inf\{\ell_F(\gamma) \mid \gamma \text{ is a non-contractible loop in } M\}.$$

One expects that the area of a Finsler manifold for which all non-contractible loops have a length uniformly bounded from below cannot be made arbitrarily small. This is described by an inequality of the form

$$\operatorname{area}(M, F) \geq C \operatorname{sys}^2(M, F)$$

holding for some set of metrics F, where C is some positive constant. Such an inequality is called an *isosystolic inequality* and the constant might depend on the set of metrics considered. Usually one considers either Riemannian metrics, reversible Finsler metrics or all Finsler metrics. An isosystolic inequality is said to be optimal if the constant C cannot be improved. Finally, it is said that there is systolic freedom if such a positive constant does not exist.

The first optimal isosystolic inequality was found for the 2-torus in 1949 by Charles Loewner. As it is explained by his student Pao Ming Pu at the end of [6], Loewner found it during the lectures of a course on Riemannian geometry he was teaching at the time. He proved that for any Riemannian metric g on the 2-torus, $\operatorname{area}(\mathbb{T}^2, g) \geq \frac{\sqrt{3}}{2} \operatorname{sys}^2(\mathbb{T}^2, g)$, and that the constant $\frac{\sqrt{3}}{2}$ is optimal. Inspired by Loewner's method, Pu proved in [6] that for the real projective plane $\operatorname{area}(\mathbb{RP}^2, g) \geq \frac{2}{\pi} \operatorname{sys}^2(\mathbb{RP}^2, g)$ for any Riemannian metric g and that the constant $\frac{2}{\pi}$ is also optimal. For the case of Finsler metrics and the 2-torus, a complete summary of optimal isosystolic inequalities is done in [2]. This article gathers all known optimal constants, including the ones for Riemannian, reversible Finsler and not-necessarily reversible Finsler metrics for both Holmes–Thompson and Busemann–Hausdorff areas. There, $\mathbb{T}^2$ is identified with the quotient of the Euclidean plane $\mathbb{R}^2$ by the integer grid $\mathbb{Z}^2$. In that case, a metric on $\mathbb{T}^2$ is just a metric on $\mathbb{R}^2$ compatible with the quotient map, and non-contractible loops in $\mathbb{T}^2$ correspond to paths between points in $\mathbb{R}^2$ that differ by some $z \in \mathbb{Z}^2 \setminus \{(0,0)\}$. The strategy followed in the article is to reduce the general case to the case where the metric is flat, in the sense that the unit balls in $T_x\mathbb{T}^2$ are the same for all $x \in \mathbb{T}^2$. Then, the inequality is most of the times a consequence of previously known results in convex geometry. See [2] for all the details.

2.1 The real projective plane

Pu, in [6], followed an analogous procedure to what Loewner did with $\mathbb{T}^2$ but for $\mathbb{RP}^2$, so it might be interesting to explicit a parallelism between $\mathbb{RP}^2$ and $\mathbb{T}^2$. What is the universal covering map of $\mathbb{RP}^2$? How can non-contractible loops in $\mathbb{RP}^2$ be characterised? Is there an analogous notion of *flat* metric for $\mathbb{RP}^2$ that makes computations easier? To answer the first question, recall that $\mathbb{RP}^2$ can be defined as a quotient space identifying antipodal points on the 2-sphere $\mathbb{S}^2$, as is shown in Figure 1. The quotient map $\mathbb{S}^2 \to \mathbb{RP}^2 \cong \mathbb{S}^2/\{\pm \mathrm{Id}\}$ is the universal covering map over $\mathbb{RP}^2$ since $\mathbb{S}^2$ is simply connected, and plays an analogous role to the quotient map $\mathbb{R}^2 \to \mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$.

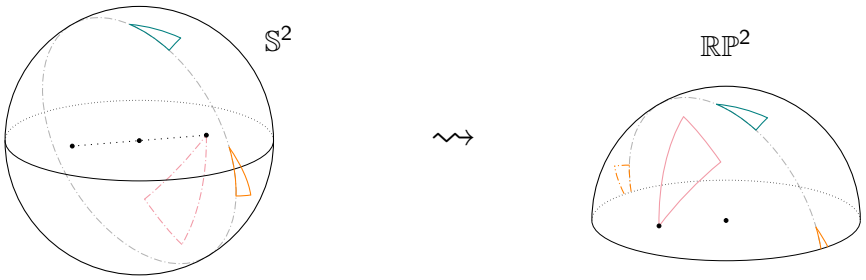


Figure 1: Universal covering map $\mathbb{S}^2 \to \mathbb{RP}^2$.

Alternatively, one could identify $\mathbb{RP}^2$ with a 2-disc $\mathbb{D}$ that has antipodal points on $\partial\mathbb{D}$ identified. When it comes to the characterisation of non-contractible loops, it can be shown that non-contractible loops in $\mathbb{RP}^2$ lift to paths in $\mathbb{S}^2$ joining antipodal points. See the illustration in Figure 2 for an intuitive idea and see, for instance, [5, Proposition 2.1] for a proof. More precisely, the condition of being non-contractible might be translated to the disc representation noting that a path in $\mathbb{S}^2$ from a point to its antipodal point must cross the horizon an odd number of times. As a subtlety, if the start and endpoints lie in the horizon, the open curve excluding these two points must cross the horizon an even number of times. Then, if starting and ending at points of the horizon counts as another cross, non-contractible loops in $\mathbb{RP}^2$ are characterised by crossing the horizon an odd number of times. Crossing the horizon is translated to jumping between opposite points of $\partial\mathbb{D}$, so non-contractible loops in $\mathbb{RP}^2$ are characterised by having an odd number of these jumps.

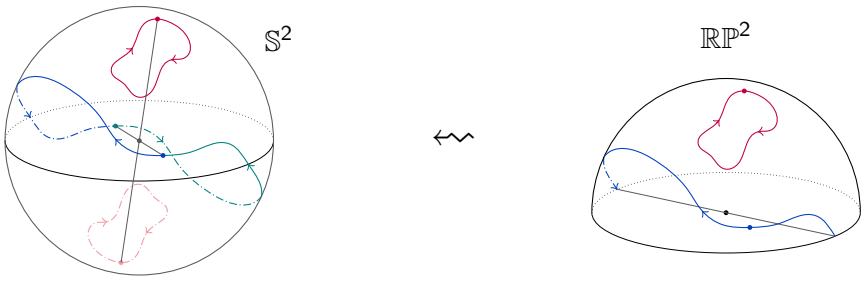


Figure 2: Correspondence between loops in $\mathbb{RP}^2$ and their lifts to $\mathbb{S}^2$.

Because translations are isometries of the Euclidean plane, a given convex body can be parallel transported from a point to another consistently to define a notion of flat Finsler metric on the 2-torus. Tangent vectors of $\mathbb{S}^2$ could also be parallel transported to another point. However, the transported vector will depend on how the parallel transport is performed. Thus, in order to get a well-defined notion of invariant metric on $\mathbb{S}^2$, one needs to assume the convex body to be rotationally invariant. In this special case, the metric is said to be a round metric on $\mathbb{RP}^2$, and can be alternatively defined as some multiple of the Riemannian metric obtained from the natural embedding of $\mathbb{S}^2$ in $\mathbb{R}^3$ as the unit Euclidean sphere. These metrics will play a similar role for $\mathbb{RP}^2$ compared to the role that flat Finsler metrics play on $\mathbb{T}^2$, although round metrics are much more restricted.

2.2 Previously known inequalities

As already mentioned, Pu proved in [6] that $\operatorname{area}(\mathbb{RP}^2, g) \geq \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, g)$ for any Riemannian metric and that equality holds if and only if g is isometric to a round metric on $\mathbb{RP}^2$. See [5, Section 4] for a proof that uses a more modern style, similarly to how the $\mathbb{T}^2$ case is treated in [2]. Note that in both cases equality holds for a flat or round metric, although for Pu's inequality all round metrics on $\mathbb{RP}^2$ are optimal while for Loewner's inequality only some flat metrics on $\mathbb{T}^2$ are optimal. In both cases, the procedure is to note that, by the uniformisation theorem, any metric is isometric to a conformal multiple of a flat or round one. Then, one observes that averaging the conformal factor gives a multiple of the flat or round metric, while it leaves the area invariant but increases the systole. Finally, the inequalities follow from the optimal flat or round metric cases. It can be computed that $\operatorname{area}(\mathbb{RP}^2, g) = \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, g)$ for any round metric on $\mathbb{RP}^2$ (see for instance [5, Section 4.2]). For the case of $\mathbb{T}^2$, before concluding, one must prove that the same isosystolic inequality holds also for any flat metric g. This is not as straightforward as for $\mathbb{RP}^2$, but it is equivalent to finding the Hermite constant γ_2, as is explained in [2].

Ivanov proved in [4] that $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) \geq \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, F)$ also holds for reversible Finsler metrics. The idea of the proof is first to consider a non-contractible loop γ_0 on $\mathbb{RP}^2$ such that $\ell_F(\gamma_0) = \operatorname{sys}(\mathbb{RP}^2, F)$, which can be done by compactness arguments. Such loops are usually called *systolic loops*. As is shown in Figure 2, the union of the two lifts of γ_0 divides the 2-sphere in two 2-discs. Considering the pullback metric φ on one of the discs $\mathbb{D}$, the inequality is reduced to finding an inequality between $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) = \operatorname{area}_{\mathrm{HT}}(\mathbb{D}, \varphi)$ and the length of $\partial\mathbb{D}$. Introducing *cyclic maps* $f = (f_1, \dots, f_n)$, Ivanov proves that

$$\operatorname{area}_{\mathrm{HT}}(\mathbb{D}, \varphi) \geq \frac{1}{2\pi}\int_{\partial\mathbb{D}} \sum_{i=1}^{n} f_i \cdot \mathrm{d}f_{i+1}. \tag{2}$$

Finally, Ivanov notes that for cyclically ordered and equidistant points $\{p_i\}_{i=1}^n \subseteq \partial\mathbb{D}$, the choice $f_i(x) = d_\varphi(p_i, x)$ leads to a *cyclic map*. See [4, Section 3] for the definition, properties and examples of *cyclic maps*. Under the assumption of a reversible metric, $\int_{\partial\mathbb{D}} f_i \cdot \mathrm{d}f_{i+1}$ is easy to compute using an arc-length parametrisation of $\partial\mathbb{D}$. In fact, it amounts to computing the signed area of the curve shown in Figure 3a. The signed area of each rectangle is $\frac{4\operatorname{sys}^2(\mathbb{RP}^2,F)}{n}\left(1-\frac{2}{n}\right)$, which leads to

$$\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) = \operatorname{area}_{\mathrm{HT}}(\mathbb{D}, \varphi) \geq \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, F)\left(1-\frac{2}{n}\right). \tag{3}$$

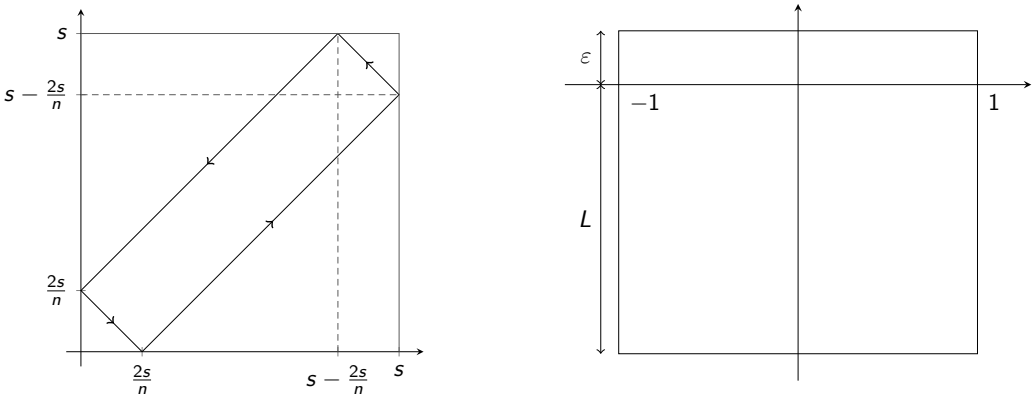
The proof is concluded noting that n can be chosen arbitrarily large. Ivanov's result and Proposition 1.2 imply that $\operatorname{area}_{\mathrm{BH}}(\mathbb{RP}^2, F) \geq \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, F)$ for any reversible Finsler metric. Note that the inequality is optimal in both cases, because equality holds for any round metric on $\mathbb{RP}^2$, which is Riemannian.

Round metrics do not seem to be relevant for Ivanov's result. Nevertheless, they play an important role in the case of $\mathbb{T}^2$. A *stable norm* on $T_x\mathbb{T}^2$, introduced in [3], is defined as $\|z\|_x = \lim_{k\to\infty}\frac{d(x,x+kz)}{k}$ for $z \in \mathbb{Z}^2$. This norm depends on the original Finsler metric on $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, and it can be shown to be independent of x. This means that the *stable metric* is flat, and it turns out that $\operatorname{area}_{\mathrm{HT}}(\mathbb{T}^2, F) \geq \operatorname{area}_{\mathrm{HT}}(\mathbb{T}^2, \|\cdot\|)$ and $\operatorname{sys}(\mathbb{T}^2, F) = \operatorname{sys}(\mathbb{T}^2, \|\cdot\|)$. Moreover, as is proven in [2], $\operatorname{area}_{\mathrm{BH}}(\mathbb{T}^2, F) \geq \operatorname{area}_{\mathrm{BH}}(\mathbb{T}^2, \|\cdot\|)$ also for reversible metrics. Thus, all these optimal isosystolic inequalities reduce to their respective flat cases. Following what is explained in [2], Minkowski's first theorem implies that $\operatorname{area}_{\mathrm{BH}}(\mathbb{T}^2, F) \geq \frac{\pi}{4}\operatorname{sys}^2(\mathbb{T}^2, F)$ for reversible and flat metrics, being optimal for the supremum norm. Due to a theorem by Mahler, the areas of a symmetric

convex ball and its dual are related by $|B_x| \cdot |B_x^\circ| \geq 8$, being also optimal for the supremum norm. This implies that $\mathrm{area}_{\mathrm{HT}}(\mathbb{T}^2, F) \geq \frac{2}{\pi}\,\mathrm{sys}^2(\mathbb{T}^2, F)$ is optimal for flat and reversible metrics. By the properties of the *stable norm* one deduces that the previous optimal inequalities for flat and reversible metrics are also valid for any reversible metrics. As a final comment, the optimal isosystolic inequalities for $\mathrm{area}_{\mathrm{HT}}(\mathbb{T}^2, F)$ and $\mathrm{area}_{\mathrm{BH}}(\mathbb{T}^2, F)$ are different for the reversible case, in contrast with the case of $\mathbb{RP}^2$. This is because, among Finsler metrics, optimal metrics F_0 for $\mathbb{T}^2$ are not Riemannian, and satisfy $\mathrm{area}_{\mathrm{BH}}(\mathbb{T}^2, F_0) > \mathrm{area}_{\mathrm{HT}}(\mathbb{T}^2, F_0)$ by Proposition 1.2, while optimal metrics for $\mathbb{RP}^2$ are the round ones, which are Riemannian.

3. Systolic freedom for Busemann–Hausdorff area

Minkowski's theorem prevents symmetric convex bodies $K \subseteq \mathbb{R}^2$ such that $\mathrm{int}(K) \cap \mathbb{Z}^2 = \{(0,0)\}$ from having a Lebesgue measure $|K| > 4$, as is explained in [2, Section 3]. The condition $\mathrm{int}(K) \cap \mathbb{Z}^2 = \{(0,0)\}$ ensures that the flat metric F with unit ball K fulfils $\mathrm{sys}(\mathbb{T}^2, F) \geq 1$. This key fact implies the optimal inequality for the Busemann–Hausdorff area and reversible metrics. However, for non-symmetric convex bodies the theorem no longer applies. In fact, as is proven in [2, Section 3.2], there exists a family of flat metrics F_ε such that $\mathrm{sys}(\mathbb{T}^2, F_\varepsilon) = 1$ and $|K_\varepsilon| = \frac{(1+\varepsilon)^2}{2\varepsilon}$ for the corresponding unit ball K_ε. By definition of the Busemann–Hausdorff area, letting $\varepsilon \to 0$ allows one to have $\mathrm{area}_{\mathrm{BH}}(\mathbb{T}^2, F_\varepsilon)$ arbitrarily small, proving systolic freedom.



(a) Here s denotes $\mathrm{sys}(\mathbb{RP}^2, F)$.

(b) The upper direction points to the pole.

Figure 3: In the left, curves in $\mathbb{R}^2$ whose signed areas give the result of the individual integrals in (2). In the right, unit balls along the meridians of the hemisphere.

For the case of $\mathbb{RP}^2$, an analogous procedure would be to look for arbitrarily large unit balls that do not lead to an arbitrarily small value for the systole. This is proven to be possible in [5, Section 6], which leads to the conclusion that systolic freedom also holds in the non-reversible case for $\mathrm{area}_{\mathrm{BH}}$. The idea behind the construction in [5] is to build a metric in a hemisphere of $\mathbb{S}^2$ such that the equator contains a systolic loop of some fixed length. In order to have a small value for $\mathrm{area}_{\mathrm{BH}}(\mathbb{RP}^2, F)$, one needs to have large unit balls in great part of the hemisphere of $\mathbb{S}^2$. However, these large unit balls (which lead to short distances) must be such that a systolic loop still lies inside the equator. This is done with unit balls of arbitrarily large size L

in one direction and arbitrarily small size ε in the opposite direction, as is shown in Figure 3b. These balls are allowed to be arbitrarily large and they prevent curves that go towards the pole from being too short. Note that they are convex sets containing the origin, so they correspond to some non-reversible Finsler metric. The final step is to make such a metric on a hemisphere of $\mathbb{S}^2$ well-defined and compatible with a metric on $\mathbb{RP}^2$. First of all, one needs to have a well-defined unit ball at the pole: it cannot depend on the meridian that approaches the point. This can be achieved changing smoothly the unit ball in Figure 3b to a rotationally invariant one around the pole. Besides, a metric F on $\mathbb{S}^2$ is compatible with a metric on $\mathbb{RP}^2$ if $F_x(u_1) = F_{-x}(u_2)$, where u_1 and u_2 are the different lifts of some $v \in T_{[x]}\mathbb{RP}^2$. Geometrically, assume that one observes $\mathbb{S}^2$ from the point such that x and $-x$ are the closest and furthest points of the equator, respectively. As is illustrated in Figure 2, from this point of view, u_1 and u_2 are half turn rotations of one another. A change in point of view so that $-x$ is now in front and still with the pole above corresponds to a horizontal flip of the view of $T_{-x}\mathbb{S}^2$. In conclusion, the unit balls of antipodal points in the equator must be vertically flipped when seen the same way as in Figure 3b. Thus, it is enough to change smoothly the unit balls in the equator to vertically symmetric ones in order to have a compatible metric.

Note that both smoothing procedures can be done without changing the Lebesgue measure of the unit balls and that the angular integration region is $\left(0, \frac{\pi}{2}\right) \times (0, 2\pi)$, which has an area of π^2. Then, for this metric, one gets from (1) that $\operatorname{area}_{\mathrm{BH}}(\mathbb{RP}^2, F) = \frac{\pi}{2(\varepsilon+L)} \cdot \pi^2$, which can be made arbitrarily small for $L \to \infty$. When it comes to the systole, recall that a lift of a non-contractible loop γ must jump between opposite points of the equator an odd number of times. Considering only a part of γ if necessary, one can assume that γ joins opposite points of the equator without any other jump in between. Note that the unit balls of Figure 3b are a non-symmetric version of the supremum norm $\|(u_1, u_2)\| = \max\{|u_1|, |u_2|\}$. For this non-symmetric version it can be computed that $\|(u_1, u_2)\| = \max\left\{|u_1|, \frac{u_2}{\varepsilon}, -\frac{u_2}{L}\right\}$. See [5, Proposition 6.2] for the details. If $\gamma = (\gamma_1, \gamma_2)$ does not enter in the smoothen zone around the pole,

$$\ell_F(\gamma) = \int_0^1 F_{\gamma(t)}(\gamma_1'(t), \gamma_2'(t))\,\mathrm{d}t \geq \left|\int_0^1 \gamma_1'(t)\,\mathrm{d}t\right| = |\gamma_1(1) - \gamma_1(0)| \geq \pi.$$

Note that equality holds if $\gamma_2'(t) = 0$ and γ_1 increases or decreases monotonically between azimuthal coordinates that differ exactly in π. If γ enters the smoothen zone around the pole, the first part of γ must join the initial point with the zone. By what has been mentioned above, the length of vectors pointing to the pole is proportional to $\frac{1}{\varepsilon}$. Then, a small enough choice of ε would imply that $\ell_F(\gamma) > \pi$ also, and therefore $\operatorname{sys}(\mathbb{RP}^2, F) = \pi$. In the end, $\operatorname{area}_{\mathrm{BH}}(\mathbb{RP}^2, F) = \frac{\pi}{2(\varepsilon+L)} \operatorname{sys}^2(\mathbb{RP}^2, F) < \frac{\pi}{2L} \operatorname{sys}^2(\mathbb{RP}^2, F)$ for any value of $L > 0$. In particular, since L can be chosen arbitrarily large, there is systolic freedom for $\mathbb{RP}^2$ and the Busemann–Hausdorff area. See [5, Section 6] for more details.

4. Optimal inequalities for non-reversible metrics

Álvarez Paiva, Balacheff and Tzanev proved in [1, Theorem IV] that $\operatorname{area}_{\mathrm{HT}}(\mathbb{T}^2, F) \geq \frac{3}{2\pi} \operatorname{sys}^2(\mathbb{T}^2, F)$ for flat metrics and that equality holds when the unit ball is the triangle with vertices $(1, 0)$, $(0, 1)$ and $(-1, -1)$. Finally, by the properties of the *stable norm*, one deduces that

$$\operatorname{area}_{\mathrm{HT}}(\mathbb{T}^2, F) \geq \operatorname{area}_{\mathrm{HT}}(\mathbb{T}^2, \|\cdot\|) \geq \frac{3}{2\pi} \operatorname{sys}^2(\mathbb{T}^2, \|\cdot\|) = \frac{3}{2\pi} \operatorname{sys}^2(\mathbb{T}^2, F)$$

also for any Finsler metric.

Finding the optimal isosystolic inequality for the more general Finsler case for $\operatorname{area}_{\mathrm{HT}}$ and $\mathbb{RP}^2$ is still an open problem. Existence of an optimal inequality can be proven by symmetrising the metric. Indeed, considering the symmetric metric $\tilde{F}_x(u) = F_x(u) + F_x(-u)$, it can be proven in dimension 2 that $|\tilde{B}_x^\circ| \leq 6|B_x^\circ|$ (see [7, Theorem 1]). If $\gamma \subseteq \mathbb{RP}^2$ is a systolic loop for $\tilde{F}$, the inverted loop $-\gamma$ is also non-contractible, and then $\operatorname{sys}(\mathbb{RP}^2, \tilde{F}) = \ell_{\tilde{F}}(\gamma) = \ell_F(\gamma) + \ell_F(-\gamma) \geq 2\operatorname{sys}(\mathbb{RP}^2, F)$. Joining these inequalities with the optimal inequality for reversible metrics,

$$\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) \geq \frac{1}{6}\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, \tilde{F}) \geq \frac{1}{6}\cdot\frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, \tilde{F}) \geq \frac{2}{3}\cdot\frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, F).$$

Note that this implies that the constant $\frac{2}{\pi}$ can be improved, at most, by a factor of $\frac{2}{3}$ for non-reversible metrics. However, [7, Theorem 2] states that $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) = \frac{1}{6}\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, \tilde{F})$ if and only if almost all unit balls are triangles. The fact that the optimal metric for the reversible case is a round one, far from having symmetrised triangular unit balls, suggests that $\frac{4}{3\pi}$ is not optimal.

Conjecture 4.1. *The optimal isosystolic inequality for Finsler metrics and Holmes–Thompson area is* $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) \geq \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, F)$.

The author has tried to attack the non-reversible case and Holmes–Thompson area with little success. Consider the family of metrics used in the proof of systolic freedom in the previous section. For simplicity, consider the metric before the smoothing, which can be done in an irrelevant arbitrarily small region. Imposing that the systole is still attained along the equator amounts to imposing that $\frac{1}{\varepsilon} + \frac{1}{L} \geq 2$. Indeed, as before, if γ does not touch the pole, $\ell_F(\gamma) \geq \pi$. And if it touches it, it must go up and then back down, having a length $\ell_F(\gamma) \geq \frac{\pi}{2}\left(\frac{1}{\varepsilon} + \frac{1}{L}\right) \geq \pi$. The dual convex body of the unit balls of Figure 3b can be computed to be the convex hull of the points $(\pm 1, 0)$, $\left(0, \frac{1}{\varepsilon}\right)$ and $\left(0, -\frac{1}{L}\right)$. This convex kite has Lebesgue measure $\frac{1}{\varepsilon} + \frac{1}{L}$, and similarly to the Busemann–Hausdorff case, by (1),

$$\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) = \frac{\frac{1}{\varepsilon} + \frac{1}{L}}{\pi}\cdot\pi^2 = \frac{1}{\pi}\left(\frac{1}{\varepsilon} + \frac{1}{L}\right)\operatorname{sys}^2(\mathbb{RP}^2, F).$$

In conclusion, $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) \geq \frac{2}{\pi}\operatorname{sys}^2(\mathbb{RP}^2, F)$ if $\frac{1}{\varepsilon} + \frac{1}{L} \geq 2$, which prevents the existence of shortcuts through the pole. The smoothening process would just lead to results arbitrarily close to the above inequality, agreeing with Conjecture 4.1.

Any unit ball can be drawn inside a rectangle and containing a triangle that touches three of the furthest points from the origin. This might leave shortest lengths invariant and it might be interesting to perform a similar test for triangle-shaped unit balls. For example, consider triangles with vertices $(1, 0)$, $(-\delta, \varepsilon)$ and $(-\delta, -L)$. In this case, the dual triangle has vertices $\left(-\frac{1}{\delta}, 0\right)$, $\left(1, \frac{1+\delta}{\varepsilon}\right)$ and $\left(1, -\frac{1+\delta}{L}\right)$, and Lebesgue measure $\frac{(1+\delta)^2}{2\delta}\left(\frac{1}{\varepsilon} + \frac{1}{L}\right)$. The norm is not so easy to compute but one could expect that imposing that the systole is attained around the equator would imply the same (or worse) inequality. It would be a surprise if there existed values for ε, L and δ that prove Conjecture 4.1 wrong. The author's search of examples that prove the conjecture wrong has been unfruitful and looking for ways to prove it might be more sensible.

A minor advance in this direction has been achieved in [5, Theorem 5.13], giving a slight generalisation of Ivanov's result for reversible metrics. It states that the inequality is also true for metrics such that the distance between any two points of a systolic loop γ_0 is attained through γ_0. In other words, one needs to have no shortcuts between points of γ_0 that deviate from γ_0. In this case, if γ_0 connects x to y (and not

the other way around), the definition of systole ensures that there are no shortcuts from x to y. However, in the non-reversible case, there might be shortcuts from y to x. Ivanov's assumption is to have a reversible metric, which implies that there are no such shortcuts. The assumption in [5, Theorem 5.13] is weaker but still ensures that there are no such shortcuts. The proof is essentially the same that the one for Ivanov's theorem although Figure 3a gets slightly modified. For instance, the curve is no longer contained in the square $[0, s]^2$, and the short straight lines become unknown but bounded. The corresponding curve is shown in [5, Figure 6], and the inequality (3) is modified to

$$\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) = \operatorname{area}_{\mathrm{HT}}(\mathbb{D}, \varphi) > \frac{2}{\pi} \operatorname{sys}^2(\mathbb{RP}^2, F) \left(\frac{n-1}{n} - 2 \cdot \frac{n-1}{n^2} \right).$$

Luckily, for arbitrarily large n the inequality becomes $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) \geq \frac{2}{\pi} \operatorname{sys}^2(\mathbb{RP}^2, F)$. A sufficient condition to avoid shortcuts is that the systolic curve γ_0 has the same forward and backward length. In particular, this holds if $F_{\gamma_0(t)}(\gamma_0'(t)) = F_{\gamma_0(t)}(-\gamma_0'(t))$ for all t. In other words, reversibility of the metric along a systolic curve is enough. Some ideas to attack the general case would be to try to modify the metric around a systolic curve to a case under which the theorem holds. This might be easier than to modify the metric at all points, although the attempts done by the author lead to inconclusive scenarios. For instance, making the unit balls symmetric along a systolic curve by enlarging them, $\operatorname{area}_{\mathrm{HT}}$ decreases but shortcuts might appear. Instead, if the balls are symmetrised by stretching them, the systole must increase, but so does the area. The only way the author has tried to define a kind of an overall averaged norm on $\mathbb{S}^2$ is considering

$$\tilde{F}_x(v) = \int_{\mathrm{SO}(3)} F_{\sigma(x)}((T_x\sigma)v) \, \mathrm{d}\mu(\sigma),$$

where μ is the unique left-invariant Haar measure on $\mathrm{SO}(3)$ such that $\mu(\mathrm{SO}(3)) = 1$. Intuitively, the unit norm has been averaged over all directions around a point and over all points, so that $\tilde{F}$ corresponds to a round metric on $\mathbb{RP}^2$. It can be proved that $\operatorname{sys}(\mathbb{RP}^2, \tilde{F}) \geq \operatorname{sys}(\mathbb{RP}^2, F)$, because any curve joining antipodal points under the action of $\sigma \in \mathrm{SO}(3)$ has the same property. However, $\operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, F) \geq \operatorname{area}_{\mathrm{HT}}(\mathbb{RP}^2, \tilde{F})$ can be false in some cases. For instance, considering the unit balls in Figure 3b, the average norm for the tangent vector $(1, 0)$ in all directions should be

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_x(\cos t, \sin t) \, \mathrm{d}t = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \max \left\{ |\cos t|, \frac{\sin t}{\varepsilon}, -\frac{\sin t}{L} \right\} \mathrm{d}t = \frac{\sqrt{1+\varepsilon^2}}{\pi\varepsilon} + \frac{\sqrt{1+L^2}}{\pi L}.$$

Then, the unit sphere is given by all vectors lying on the Euclidean circle with radius $r = \frac{\pi}{\frac{\sqrt{1+\varepsilon^2}}{\varepsilon} + \frac{\sqrt{1+L^2}}{L}}$. For the case of $\varepsilon = L = 1$, recall that $|B_x^\circ| = \frac{1}{\varepsilon} + \frac{1}{L} = 2$, and for the averaged metric,

$$|\tilde{B}_x^\circ| = \frac{\pi}{r^2} = \frac{1}{\pi} \left(\frac{\sqrt{1+\varepsilon^2}}{\varepsilon} + \frac{\sqrt{1+L^2}}{L} \right)^2 = \frac{(\sqrt{2}+\sqrt{2})^2}{\pi} = \frac{8}{\pi} > |B_x^\circ|.$$

This shows that the averaging procedure fails to have good properties even for the supremum norm. As was suggested by F. Balacheff, another approach could be to consider a contact structure on the unitary tangent bundle $S^*\mathbb{RP}^2$. With contact forms there is a theorem similar to the uniformisation theorem that says that the initial contact form and a fixed round one are contactomorphic. One might be able to average over the group of diffeomorphisms of $S^*\mathbb{RP}^2$ that leaves the round contact form invariant. This is similar

to the fact that the action of SO(3) leaves a round metric on $\mathbb{S}^2$ invariant. It turns out that $S^*\mathbb{RP}^2$ is isomorphic to the Lens space $L(4,1)$. However, the systole seems to be more difficult to deal with.

A final idea to believe that Conjecture 4.1 is true is the following. Consider an attempt of minimising the Holmes–Thompson area only around a systolic loop with a fixed length. In order to decrease the value of area_{HT} one must increase the Lebesgue measure of the unit balls. However, this process could be intuitively done until the metric is symmetric along the systolic loop because otherwise the systole might decrease. In conclusion, it seems sensible that the metric that minimises area_{HT} is symmetric along a systolic loop, and the generalisation of Ivanov's theorem would apply in this case.

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Properties of triangular partitions and their generalizations

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Resum *(CAT)*

Una partició entera es diu triangular si el seu diagrama de Ferrers es pot separar del seu complement (com a subconjunt de $\mathbb{N}^2$) amb una línia recta. Aquest article es basa en alguns desenvolupaments recents sobre el tema per derivar noves propietats enumeratives, geomètriques i algorísmiques d'aquests objectes. La investigació s'estén després a generalitzacions en dimensions superiors, anomenades particions piramidals, i a particions convexes i còncaves, definides com particions amb un diagrama de Ferrers que pot ser separat del seu complement per una corba convexa o còncava.

Abstract *(ENG)*

An integer partition is said to be triangular if its Ferrers diagram can be separated from its complement (as a subset of $\mathbb{N}^2$) by a straight line. This article builds on some recent developments on the topic in order to derive new enumerative, geometric and algorithmic properties of these objects. The research is then extended to higher-dimensional generalizations, called pyramidal partitions, and to convex and concave partitions, defined as partitions whose Ferrers diagram can be separated from its complement by a convex or concave curve.

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1. Introduction

An integer partition is said to be triangular if its Ferrers diagram can be separated from its complement by a straight line. These objects first appeared in the contexts of combinatorial number theory [3] and computer vision [4]. From a combinatorial perspective, they were first studied by Onn and Sturmfels [11], who defined them in any dimension and called them *corner cuts*. Shortly after, Corteel et al. [5] obtained the generating function for the number of 2-dimensional corner cuts. More recently, triangular partitions have attracted interest in the field of algebraic combinatorics. Motivated by work of Blasiak et al. [2] generalizing the shuffle theorem for paths under a line, Bergeron and Mazin [1] coined the term *triangular partitions* and studied some of their combinatorial properties.

In this article we present new enumerative, geometric and algorithmic properties of triangular partitions and their generalizations. In Section 2 we give basic definitions and some results from [1, 5]. In Section 3 we introduce a natural alternative characterization of triangular partitions, as those such that the convex hull of the Ferrers diagram and that of its complement do not intersect. Moreover, we characterize which points may be added to or removed from the Ferrers diagram while preserving triangularity.

In Section 4, we present two ways to encode triangular partitions in terms of balanced words, and use one of them to implement an algorithm which, for a given N, computes the number of triangular partitions of size $n \leq N$ in time $\mathcal{O}(N^{5/2})$. This allows us to obtain the first 10^5 terms of this sequence, while just 39 terms were known previously.

In Section 5, refining the approach from [5], we obtain generating functions for triangular partitions with a given number of removable and addable cells. In Section 6, we present a recurrence for the number of triangular partitions contained in a fixed triangular partition, as well as an explicit formula involving Euler's totient function for the case where the fixed partition is a staircase. A new combinatorial proof of Lipatov's enumeration theorem for balanced words [8] is obtained as a byproduct.

Section 7 studies pyramidal partitions, which are an extension of triangular partitions to higher dimensions. We prove that the characterization in terms of convex hulls generalizes nicely and that, for dimension 3 or higher, the number of removable and addable cells can be arbitrarily large. We also describe the residue modulo d of the number of d-dimensional pyramidal partitions of size n, for d prime.

In Section 8, convex and concave partitions are analyzed. These are partitions whose Ferrers diagram can be separated from its complement by a convex or concave line. We present several characterizations and we describe their removable and addable cells in terms of convex hulls. Finally, we prove that there exist constants a, b, c such that the number of convex partitions of size n is greater than $\exp(a\sqrt[3]{n})$ and smaller than $\exp(b\sqrt[3]{n}\log n)$, and the number of concave partitions of size n is greater than $\exp(c\sqrt[3]{n})$.

Due to space constraints, proofs are omitted from this article. A more thorough explanation of the results is detailed by Elizalde and the present author in [7].

2. Background

A *partition* λ is a weakly decreasing sequence of positive integers, called the *parts* of λ. We will denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, or $\lambda = \lambda_1\lambda_2 \ldots \lambda_k$ when there is no possibility of confusion. We call $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ the *size* of λ. If $|\lambda| = n$, we say that λ is a partition of n.

Let $\mathbb{N}$ denote the set of positive integers. The *Ferrers diagram* of λ is the set of lattice points

$$\{(a, b) \in \mathbb{N}^2 \mid 1 \leq b \leq k,\ 1 \leq a \leq \lambda_b\}.$$

We will often identify a lattice point (a, b) with the unit square (called a *cell*) whose north-east corner is (a, b). In particular, we say that a cell lies above, below or on a line when the north-east corner does. The Ferrers diagram can then be interpreted as a set of cells. We will often identify λ with its Ferrers diagram, and use notation such as $c = (a, b) \in \lambda$.

Let $\sigma^k = (k, k-1, \ldots, 2, 1)$ denote the *staircase partition* of k parts. The *conjugate* λ' of λ is obtained by reflecting its Ferrers diagram about the $y = x$ axis. The *complement* of λ is defined to be the set $\mathbb{N}^2 \setminus \lambda$, where λ is identified with its Ferrers diagram.

Definition 2.1. A partition τ is *triangular* if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below the line that passes through $(0, s)$ and $(r, 0)$ for some $r, s \in \mathbb{R}_{>0}$, called a *cutting line*.

See the left of Figure 1 for an example. We often use τ to denote a triangular partition.

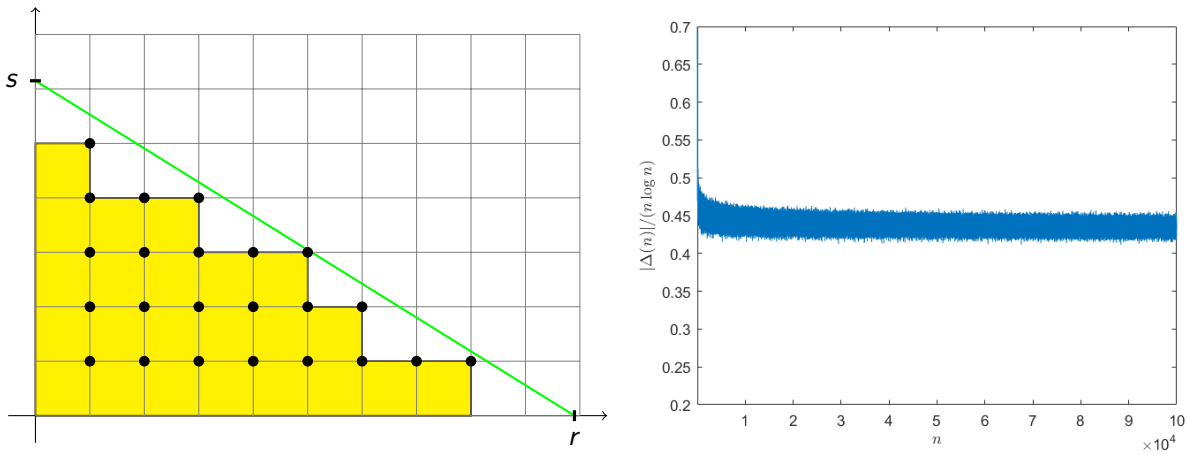

Figure 1: Left: A cutting line for the triangular partition $(8, 6, 5, 3, 1)$. Right: The first 10^5 terms of the sequence $|\Delta(n)|/(n \log n)$.

Denote by Δ the set of all triangular partitions and by $\Delta(n)$ the set of triangular partitions of size n. Corteel et al. [5] obtain the generating function of $|\Delta(n)|$ and bound the asymptotic growth of this number.

Theorem 2.2 ([5]). *The generating function for triangular partitions can be expressed as*

$$G_\Delta(z) = \sum_{n \geq 0} |\Delta(n)| z^n = \frac{1}{1-z} + \sum_{\gcd(a,b)=1} \sum_{\substack{0 \leq j < a \\ 0 \leq i < b}} \sum_{1 \leq m < k} z^{N_\Delta(a,b,k,m,i,j)},$$

where

$$\begin{aligned} N_\Delta(a, b, k, m, i, j) = (k-1)\left(\frac{(a+1)(b+1)}{2} - 1\right) + \binom{k-1}{2} ab + ij \\ + i(k-1)a + j(k-1)b + T(a, b, j) + T(b, a, i) + m, \end{aligned} \tag{1}$$

and $T(a, b, j) = \sum_{r=1}^{j} (\lfloor rb/a \rfloor + 1)$.

Theorem 2.3 ([5]). *There exist positive constants c and c' such that, for all $n > 1$,*

$$cn \log n < |\Delta(n)| < c' n \log n.$$

Let $c = (i, j)$ be a cell of a triangular partition $\lambda = \lambda_1 \dots \lambda_k$. Define the *arm length* and the *leg length* of c to be $a(c) = \lambda_j - i$ and $\ell(c) = \lambda'_i - j$, that is, the number of cells to the right of c in its row, and above c in its column, respectively. Bergeron and Mazin [1] characterize triangular partitions and study the number of cells that can be added or removed while preserving triangularity.

Lemma 2.4 ([1, Lemma 1.2]). *A partition λ is triangular if and only if $t_\lambda^- < t_\lambda^+$, where*

$$t_\lambda^- = \max_{c \in \lambda} \frac{\ell(c)}{a(c) + \ell(c) + 1}, \quad \text{and} \quad t_\lambda^+ = \min_{c \in \lambda} \frac{\ell(c) + 1}{a(c) + \ell(c) + 1}.$$

Definition 2.5. A cell of $\tau \in \Delta$ is *removable* if removing it from τ yields a triangular partition. A cell of the complement $\mathbb{N}^2 \setminus \tau$ is *addable* if adding it to τ yields a triangular partition.

Lemma 2.6 ([1, Lemma 4.5]). *Every nonempty triangular partition has either one removable cell and two addable cells, two removable cells and one addable cell, or two removable cells and two addable cells.*

3. Characterization of triangular partitions

In this section, we introduce a new characterization of triangular partitions in terms of convex hulls. This characterization is natural and arguably simpler than the one given in Lemma 2.4 by Bergeron and Mazin [1], which involves the computation of an expression in terms of arm and leg lengths for each cell. We also present a way to identify removable and addable cells. The convex hull of a set $S \subseteq \mathbb{N}^2$ will be denoted by $\mathrm{Conv}(S)$.

Proposition 3.1. *A partition λ is triangular if and only if* $\mathrm{Conv}(\lambda) \cap \mathrm{Conv}(\mathbb{N}^2 \setminus \lambda) = \emptyset$.

We will use the term *vertex* to refer to a 0-dimensional face of a polygon; in particular, not all lattice points of $\mathrm{Conv}(\tau)$ are vertices.

Proposition 3.2. *Two cells in $\tau \in \Delta$ are removable if and only if they are consecutive vertices of* $\mathrm{Conv}(\tau)$ *and the line passing through them does not intersect* $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$. *Similarly, two cells in $\mathbb{N} \setminus \tau$ are addable if and only if they are consecutive vertices of* $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$ *and the line passing through them does not intersect* $\mathrm{Conv}(\tau)$.

An immediate corollary is that a triangular partition cannot have more than two removable cells and two addable cells, as we know from Lemma 2.6 by Bergeron and Mazin [1].

A similar characterization in terms of convex hulls for a single removable cell is proved by Elizalde and the present author in [7], and is then used to describe an algorithm that determines whether a partition λ of n into k parts is triangular. Said algorithm has complexity $\mathcal{O}(k)$ for the initialization and $\mathcal{O}(\min\{k, \sqrt{n}\})$ for the rest of its steps, whereas an algorithm based on Bergeron and Mazin's Lemma 2.4 would take time $\mathcal{O}(n)$.

4. Bijections to balanced words and efficient generation

In this section, we present two different interpretations of triangular partitions in terms of finite Sturmian words, also known as balanced words. The first interpretation, which is hinted at in [1], is quite natural, and it will allow us to prove an enumeration formula in Section 6. The second one relates each triangular partition to a balanced word together with two positive integers, and it will be used in Section 4.4 to implement an efficient algorithm to count triangular partitions by size.

4.1 Background on balanced words

A finite consecutive subword of a word is called a *factor*. An infinite binary word s is *Sturmian* if, for every $\ell \geq 1$, the number of factors of s of length ℓ is exactly $\ell + 1$. The applications of Sturmian words range from combinatorics and number theory to dynamical systems; see [9] for a thorough study.

A finite binary word $w = w_1 \dots w_\ell$ is a factor of some Sturmian word if and only if it is *balanced*, that is, for any positive integers $h \leq \ell$ and $i, j \leq \ell - h + 1$, we have

$$|(w_i + w_{i+1} + \cdots + w_{i+k-1}) - (w_j + w_{j+1} + \cdots + w_{j+k-1})| \leq 1.$$

This condition states that for any two factors of w of the same length, the number of ones in these factors differs by at most 1. Denote by $\mathcal{B}$ the set of all balanced words, and by $\mathcal{B}_\ell$ the set of those of length ℓ.

The following enumeration formula for balanced words was first proved by Lipatov [8]. Let φ denote Euler's totient function.

Theorem 4.1 ([8]). *The number of balanced words of length ℓ is*

$$|\mathcal{B}_\ell| = 1 + \sum_{i=1}^{\ell} (\ell - i + 1)\varphi(i).$$

4.2 First Sturmian interpretation

Definition 4.2. A triangular partition is *wide* if all its parts are distinct. A partition is *tall* if its conjugate is wide.

It can be shown that every triangular partition must be wide or tall, and it is both wide and tall if and only if it is a staircase. The following proposition is a consequence of a well-known bijection between balanced words and lattice paths with steps in $\{(1,0),(1,1)\}$ (see [9]).

Given a wide triangular partition $\tau = \tau_1 \dots \tau_k$, define the binary word

$$\omega(\tau) = 10^{\tau_1 - \tau_2 - 1} 10^{\tau_2 - \tau_3 - 1} \dots 10^{\tau_{k-1} - \tau_k - 1} 10^{\tau_k - 1}. \tag{2}$$

Since τ is wide, the exponents are nonnegative. For example, $\omega(86531) = 10110101$.

Proposition 4.3. *For every $k, \ell \geq 1$, the map ω is a bijection between the set of wide triangular partitions with k parts and first part equal to ℓ, and the set of balanced words of length ℓ with k ones that start with 1.*

4.3 Second Sturmian interpretation

To our knowledge, our second encoding of triangular partitions using balanced words is new. Let $\mathcal{W}$ be the set of wide triangular partitions with at least two parts, and let $\mathcal{B}^0$ denote the set of balanced words that contain at least one 0.

First we describe the set of differences of consecutive parts in a wide triangular partition. For $\tau = \tau_1 \dots \tau_k \in \mathcal{W}$, define

$$\mathcal{D}(\tau) = \{\tau_1 - \tau_2, \tau_2 - \tau_3, \dots, \tau_{k-1} - \tau_k\}.$$

Lemma 4.4. *For any $\tau = \tau_1 \dots \tau_k \in \mathcal{W}$, there exists $d \in \mathbb{N}$ such that $\tau_k \leq d+1$ and either $\mathcal{D}(\tau) = \{d\}$ or $\mathcal{D}(\tau) = \{d, d+1\}$.*

Define $\min(\tau) = \tau_k$, $\operatorname{dif}(\tau) = \min \mathcal{D}(\tau)$, and $\operatorname{wrd}(\tau) = w_1 \dots w_{k-1}$, where, for $i \in [k-1]$, we let $w_i = \tau_i - \tau_{i+1} - \operatorname{dif}(\tau)$. Lemma 4.4 guarantees that $\operatorname{wrd}(\tau)$ is a binary word.

Theorem 4.5. *The map $\chi = (\min, \operatorname{dif}, \operatorname{wrd})$ is a bijection between $\mathcal{W}$ and the set*

$$\mathcal{T} = \{(m, d, w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}^0 \mid m \leq d+1;\ w1 \in \mathcal{B}^0 \text{ if } m = d+1\}.$$

Its inverse is given by the map

$$\xi(m, d, w_1 \dots w_{k-1}) = \tau_1 \dots \tau_k, \quad \text{where } \tau_i = m + \sum_{j=i}^{k-1} (w_j + d) \text{ for } i \in [k].$$

Additionally, given $\tau \in \mathcal{W}$ with image $\chi(\tau) = (m, d, w)$, its number of parts equals the length of w plus one, and its size is

$$|\tau| = km + \binom{k}{2} d + \sum_{i=1}^{k-1} i w_i. \tag{3}$$

4.4 Efficient generation

Before this work, the entry of the OEIS [10, A352882] for the number triangular partitions of n only included values for $n \leq 39$. These are the terms listed in [5], where they are obtained from the generating function in Theorem 2.2. This approach turns out to be impractical for large n.

Theorem 4.5 can be used to implement a much more efficient algorithm that can quickly compute the first 10^5 terms of the sequence. Consider the tree where each vertex is a balanced word of length at most $\lfloor \sqrt{2N} \rfloor$, and the parent of a nonempty word is the word obtained by removing its last letter. On input N, our algorithm runs a depth first search through this tree.

For each $w \in \mathcal{B}_\ell$ with $\ell \leq \sqrt{2N}$, the algorithm finds all the values $m, d \in \mathbb{N}$ such that $(m, d, w) \in \mathcal{T}$, as defined in Theorem 4.5, and such that the size function given in equation (3) is at most N. Each triplet (m, d, w) corresponds to two partitions, the wide triangular partition $\tau = \chi(m, d, w)$ and its conjugate, except when $w = 0^{k-1}$ (for some $k \geq 2$) and $m = d$, in which case it accounts for only one partition, the staircase σ^k.

A C++ implementation of this algorithm can be found at [6]. In a standard laptop computer, this algorithm generates the first 10^3 terms of the sequence $|\Delta(n)|$ in under one second, the first 10^4 terms in under ten seconds, and the first 10^5 terms in under one hour.

Proposition 4.6. *The above algorithm finds $|\Delta(n)|$ for $1 \leq n \leq N$ in time $\mathcal{O}(N^{5/2})$. Additionally, it can be modified to generate all (resp. all wide) triangular partitions of size at most N in time $\mathcal{O}(N^3 \log N)$ (resp. $\mathcal{O}(N^{5/2} \log N)$).*

The plot on the right of Figure 1 portrays the first 10^5 terms of the sequence $|\Delta(n)|/(n \log n)$. A qualitative study suggests that, for large n, this sequence oscillates between two decreasing functions that differ by about 0.05.

5. Generating functions for subsets of triangular partitions

Let Δ_1 and Δ_2 denote the subsets of triangular partitions with one removable cell and with two removable cells, respectively. Let Δ^1 and Δ^2 denote the subsets of triangular partitions with one addable cell and with two addable cells, respectively. Let $\Delta_2^2 = \Delta_2 \cap \Delta^2$. Denote partitions of size n in each subset by $\Delta_1(n)$, $\Delta_2(n)$, $\Delta^1(n)$, $\Delta^2(n)$ and $\Delta_2^2(n)$. In this section we obtain generating functions for each of these sets, refining Theorem 2.2. In the following proposition, $N_\Delta(a, b, k, m, i, j)$ is the function defined in equation (1).

Proposition 5.1. *The generating function for triangular partitions with two removable cells can be expressed as*

$$G_{\Delta_2}(z) = \sum_{n\geq 0} |\Delta_2(n)| z^n = \sum_{\gcd(a,b)=1} \sum_{\substack{0\leq j<a \\ 0\leq i<b}} \sum_{k\geq 2} z^{N_\Delta(a,b,k,k,i,j)}.$$

Proposition 5.2. *The generating functions for partitions in Δ_1, Δ^2, Δ^1, Δ_2^2 can be written in terms of $G_\Delta(z)$ (given in Theorem 2.2) and $G_{\Delta_2}(z)$ (given in Proposition 5.1) as follows:*

$$G_{\Delta_1}(z) = G_\Delta(z) - G_{\Delta_2}(z) - 1, \qquad G_{\Delta^2}(z) = \frac{1-z}{z} G_\Delta(z) + \frac{1}{z} G_{\Delta_2}(z) - \frac{1}{z},$$

$$G_{\Delta^1}(z) = \frac{2z-1}{z} G_\Delta(z) - \frac{1}{z} G_{\Delta_2}(z) + \frac{1}{z}, \qquad G_{\Delta_2^2}(z) = \frac{1-2z}{z} G_\Delta(z) + \frac{1+z}{z} G_{\Delta_2}(z) - \frac{1}{z}.$$

We have used Proposition 5.1 in order to implement an algorithm to find $|\Delta_2(n)|$, available at [6]. The initial terms of the sequences $|\Delta_1(n)|$ and $|\Delta_2(n)|$ suggest that $|\Delta_2(n)| > |\Delta_1(n)|$ for all $n \geq 9$, although we do not have a proof of this. It is interesting to note that, at least for $n \leq 150$, both the local maxima of $|\Delta_1(n)|$ and the local minima of $|\Delta_2(n)|$ occur precisely when $n \equiv 2 \pmod 3$. On the other hand, $|\Delta(n)|$ does not show such periodic extrema.

6. Triangular subpartitions and a combinatorial proof of Lipatov's formula for balanced words

Let $I(\tau) = |\{\zeta \in \Delta : \zeta \subseteq \tau\}|$ denote the number of triangular subpartitions of $\tau \in \Delta$. We start by giving a recurrence for this number. In the case where τ is a staircase, we obtain an explicit formula too, deriving a new proof of Theorem 4.1 in the process.

Let c^- and c^+ be the leftmost and rightmost removable cells of τ. Following the notation in [1], let τ° be the triangular partition obtained from τ by removing all the cells in the segment between c^- and c^+ (or, if $c^- = c^+$, just removing that cell).

Lemma 6.1. *For any $\tau \in \Delta(n)$ with $n \geq 1$,*

$$I(\tau) = I(\tau \setminus \{c^-\}) + I(\tau \setminus \{c^+\}) - I(\tau^\circ) + 1.$$

This recurrence relation comes from an inclusion-exclusion argument. Along with the base case $I(\epsilon) = 1$ (where ϵ denotes the empty partition), it allows us to compute $I(\tau)$ for any $\tau \in \Delta$, although not very efficiently. We will now present a more convenient formula for the case in which τ is a staircase.

We use the terms *height* and *width* of a partition τ to refer to the number of parts and the largest part of τ, respectively. Let $\Delta^{\ell\times\ell}$ be the set of triangular partitions whose width and height are at most ℓ. It can be proved that a partition belongs to $\Delta^{\ell\times\ell}$ if and only if it is a triangular subpartition of σ^ℓ. Our next goal is to give a formula for $I(\sigma^\ell) = |\Delta^{\ell\times\ell}|$. The proof of the following lemma uses the bijection ω from equation (2).

Lemma 6.2. *For $\ell \geq 1$, the number of triangular partitions of width exactly ℓ and height at most ℓ is $|\mathcal{B}_\ell|/2$, and*

$$|\Delta^{\ell\times\ell} \setminus \Delta^{(\ell-1)\times(\ell-1)}| = I(\sigma^\ell) - I(\sigma^{\ell-1}) = |\mathcal{B}_\ell| - 1.$$

Combining the above lemma with Lipatov's Theorem 4.1 enumerating balanced words, we deduce the following result.

Theorem 6.3. *For any $\ell \geq 0$,*

$$|\Delta^{\ell\times\ell}| = I(\sigma^\ell) = 1 + \sum_{i=1}^{\ell} \binom{\ell - i + 2}{2} \varphi(i).$$

Unfortunately, the proof of Theorem 6.3 using Lemma 6.2 and Lipatov's formula does not give a conceptual understanding of why the terms $\binom{\ell-i+2}{2}$ and $\varphi(i)$ appear.

Instead, we have been able to find a direct combinatorial proof of Theorem 6.3 that explains the role of these terms. Since the whole proof does not fit in this article, we will briefly outline its main ideas. First, we establish a bijection ϕ between triangular partitions that contain the cell $(2, 1)$ and the set $\{(a, b, d, e) \in \mathbb{N}^4 \mid d < a,\ \gcd(d, e) = 1\}$, and characterize the image of $\Delta^{\ell\times\ell}$ by ϕ. Then, for a fixed pair of coprime numbers $d < e$, we take the union of the points (a, b) for which $(a, b, d, e) \in \phi(\Delta^{\ell\times\ell})$ and an affine transformation of the points (a, b) for which $(a, b, e, e - d) \in \phi(\Delta^{\ell\times\ell})$. The resulting set is formed by the lattice points inside a certain triangle, which are counted by $\binom{\ell-e+2}{2}$. Summing over all coprime pairs $d < e$ and taking into account some technical details, we obtain the formula in Theorem 6.3.

As an added benefit, our argument also provides a new proof of Lipatov's formula (Theorem 4.1).

7. Pyramidal partitions

In this section, we will study a higher-dimensional analogue of triangular partitions. These objects are first defined in [11], and some bounds on their growth are given in [13].

Definition 7.1. A d-dimensional pyramidal partition is a finite set of points in $\mathbb{N}^d$ that can be separated from its complement by a hyperplane.

Notice that a 2-dimensional pyramidal partition is the Ferrers diagram of a triangular partition. Proposition 3.1 can be extended to this more general setting; however, Lemma 2.6 does not hold anymore.

Theorem 7.2. *Let $d \in \mathbb{N}$. A finite nonempty subset $\pi \subset \mathbb{N}^d$ is a d-dimensional pyramidal partition if and only if* $\mathrm{Conv}(\pi) \cap \mathrm{Conv}(\mathbb{N}^d \setminus \pi) = \emptyset$.

Proposition 7.3. *For any $d \geq 3$, there are d-dimensional pyramidal partitions with an arbitrarily large number of removable and addable cells.*

In the case of triangular partitions in $\mathbb{N}^2$, we have that the only partitions $\tau \in \Delta$ such that $\tau = \tau'$ (that is, they are symmetrical with respect to the line $x = y$) are the staircase partitions. From this fact, we can deduce that $|\Delta(n)| \equiv 1 \pmod 2$ when $n = \binom{m}{2}$ for some integer $m \geq 2$, and $|\Delta(n)| \equiv 0 \pmod 2$ otherwise. This approach can be extended to d-dimensional pyramidal partitions by studying an action of the symmetric group on them. We will denote by $\Delta_{d\mathrm{D}}(n)$ the set of d-dimensional pyramidal partitions of size n, to avoid confusion with $\Delta_1(n)$ and $\Delta_2(n)$ defined in Section 5.

Theorem 7.4. *Let $n, d \in \mathbb{N}$, with d a prime number. If there exists an integer $m \geq d$ such that $n = \binom{m}{d}$, then $|\Delta_{d\mathrm{D}}(n)| \equiv 1 \pmod d$. Otherwise, $|\Delta_{d\mathrm{D}}(n)| \equiv 0 \pmod d$.*

8. Convex and concave partitions

Convex partitions are defined by Dean Hickerson in [10, A074658], where the number of convex partitions of size n is counted for $n \leq 55$. The concept of concave partitions is essential to some Schur positivity conjectures (see [2, Conjecture 7.1.1]). In this section, we will extend our research on triangular partitions to these more general families, starting with some characterizations.

Definition 8.1. A partition λ is said to be *convex* (resp. *concave*) if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below some convex (resp. concave) curve.

Proposition 8.2. *Given a partition λ, the following are equivalent:*

1. *λ is convex (resp. concave).*
2. *λ can be obtained as the intersection (resp. union) of a finite number of triangular partitions.*
3. $\mathrm{Conv}(\lambda) \cap (\mathbb{N}^2 \setminus \lambda) = \emptyset$ *(resp.* $\lambda \cap \mathrm{Conv}(\mathbb{N}^2 \setminus \lambda) = \emptyset$*).*
4. *There exists a convex (resp. concave) region $R \subset \mathbb{R}^2_{\geq 0}$ such that $\lambda = R \cap \mathbb{N}^2$.*

Using these new concepts, we can give a new characterization for triangular partitions.

Corollary 8.3. *A partition is triangular if and only if it is convex and concave.*

However, this characterization does not generalize to higher dimensions (see [12]).

Removable and addable cells in the convex and concave settings are defined in an analogous way to Definition 2.5.

Proposition 8.4. *A cell $c = (a, b)$ is removable from a convex partition η if and only if it is a vertex of* $\mathrm{Conv}(\eta)$ *and $(a+1, b), (a, b+1) \notin \eta$. Similarly, a cell c' is addable to a concave partition ν if and only if it is a vertex of* $\mathrm{Conv}(\mathbb{N}^2 \setminus \nu)$.

To close the article, we will study the asymptotic growth of the number of convex or concave partitions. We will use $\bigcap(n)$ (resp. $\bigcup(n)$) for the set of convex (resp. concave) partitions of size n.

Theorem 8.5. *There exists a constant b and a function $\delta(n) \sim \frac{3^{2/3}}{2} n^{2/3}$ such that*

$$\frac{2\sqrt[3]{n}\sqrt{2\sqrt[3]{n}-2}}{4\sqrt[3]{n}+4} \leq \left|\bigcap(n)\right| \leq \exp(b\sqrt[3]{n}\log n), \qquad \frac{2\sqrt[3]{4(n-\delta(n))}}{\sqrt{2+2\sqrt[3]{4(n-\delta(n))}}} \leq \left|\bigcup(n)\right|.$$

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Algebraic topology of finite topological spaces

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Aquest treball segueix les línies d'estudi de R. Stong i M. McCord dels espais topològics finits. Tot i que són dues aproximacions diferents, tenen un punt de contacte: els posets. Per una banda, classificarem els espais topològics finits a través dels posets, segons el teorema de classificació de Stong. Per l'altra, veurem com els posets codifiquen tant la informació homotòpica d'un poliedre com la d'un espai topològic finit, seguint el teorema de McCord. Conclourem el treball donant un model finit d'una superfície compacta connexa.

Abstract *(ENG)*

This work follows R. Stong and M. McCord's study lines on finite topological spaces. Although they are two different approaches, they intersect at one point: posets. On the one hand, we will classify finite topological spaces through posets, according to Stong's Classification Theorem. On the other hand, following McCord's Theorem, we will examine how posets encode the homotopic information of both polyhedra and finite topological spaces. We will conclude by providing a finite model of a compact connected surface.

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1. Introduction

A finite topological space is a topological space that has a finite number of points. At first, one might think that these topological spaces are not very interesting, cannot generate many topologies, and that the homotopy groups vanish immediately. However, they have more structure since every topological space can be associated with a partial order. Partially ordered sets, posets for short, are potent combinatorial objects for encoding information about spaces. These, in turn, are related to simplicial complexes.

Michael McCord and Robert Stong were two American mathematicians from the second half of the 20th century who studied finite topological spaces almost simultaneously in 1966, but from two different perspectives. In this work, we study finite topological spaces following these two classical approaches, to see how they relate to each other through posets. Stong's Classification Theorem [6] is based on the internal structure of the spaces, the order structure. On the other hand, McCord's Theorem [4] compares finite spaces with simplicial complexes through homotopy theory, from a more external point of view. Since this document is intended for general audiences, it does not include proofs. However, if the reader is curious, they can consult the references for details.

2. Preliminaries

In this section, we will see that finite topological spaces and finite posets are essentially the same. We will look at basic definitions for working with finite topological spaces and review some properties.

Definition 2.1. A *finite topological space* (X, τ) is a topological space over a finite set of points.

One might think that in a finite set there is a finite number of topologies and that this fact would suffice for classification. However, the notion of homeomorphism is too restrictive, and we aim to understand these spaces using their topological properties concerning homotopy properties. To grasp some concepts, it is recommended to have a basic understanding of homotopy theory.

2.1 Properties

The Alexandroff topology [1] is characterized by the property that the intersection of any family of open sets is open. Finite topological spaces exemplify this topology, since the arbitrary intersection of open sets cannot be infinite. Consequently, we can talk about the smallest open set containing a point, in the sense of an open closure.

Definition 2.2. Given a point x in a finite topological space (X, τ), we define the *minimal open set of* x as the intersection of all open sets containing x:

$$U_x = \bigcap_{x \in \mathcal{O} \in \tau} \mathcal{O}.$$

The minimal open sets form a basis for the topology of X, called the *minimal basis of* X.

Definition 2.3. A *preorder* is a reflexive and transitive relation. A *preordered set* or *preset* is a set with a preorder. A *partial order* is a reflexive, transitive and antisymmetric relation. A *partially ordered set* or *poset* is a set with a partial order.

Proposition 2.4. *Let X be a finite topological space. The binary relation $\leq$ on X defined by the following expression is a preorder:*

$$x \leq y \text{ if and only if } x \in U_y.$$

We have just seen that finite spaces induce a preorder, given by the minimal basis. Now we will see that, in fact, a preorder also induces a topology on a finite set.

Definition 2.5. Given P a preset and $x \in P$, we write $P_{\leq x} := \{z \in P \mid z \leq x\}$. Similarly, $P_{\geq}$, $P_{<}$, and $P_{>}$ are defined.

In the case of a finite topological space X with associated preorder $\leq$, U_x corresponds to $P_{\leq x}$.

Definition 2.6. Let P be a finite preset. The *Alexandroff topology* is the topology defined by the basis

$$\{P_{\leq x} \subseteq P \mid x \in P\}.$$

In fact, these two are equivalent.

Proposition 2.7 ([7, Proposition 2.1.7])**.** *Let $(X, \leq)$ be a preset and τ the Alexandroff topology. Let $\leq'$ be the preorder on X given by the minimal open sets of (X, τ). Then, the two presets $(X, \leq)$ and $(X, \leq')$ coincide.*

Proposition 2.8 ([2, Proposition 1.2.1])**.** *A function $f : X \to Y$ between finite spaces is continuous if and only if it is order-preserving.*

Definition 2.9. A topological space X satisfies the *separation axiom* T_0 if, given two distinct points, there is an open set containing one of them but not the other.

Proposition 2.10 ([7, Proposition 2.1.9])**.** *A finite topological space X is T_0 if and only if its associated preordered set is antisymmetric; therefore, it is a poset.*

We have seen that the correspondence between finite topological spaces and preorders is bijective, and, in fact, if the topological space is T_0, we have antisymmetry and hence a partially ordered set. One of the main consequences of this correspondence is the visual representation that arises: Hasse diagrams.

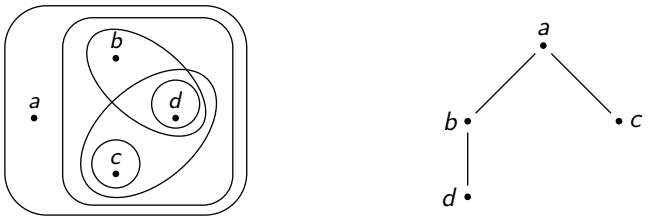
Definition 2.11. The *Hasse diagram* of a poset X is a directed graph whose vertices are the points of X and whose edges are the ordered pairs (x, y) such that $x < y$ and there exists no $z \in X$ such that $x < z < y$. Additionally, the elements are arranged in descending order, with bigger elements in the upper part of the diagram, while smaller ones are placed below.

Given any finite topological space, we can construct a T_0 space that is homotopy equivalent to the given one, by identifying points with the same closure (see [2, Proposition 1.3.1]). We will now study finite topological spaces equivalent under homotopies, therefore, without loss of generality, we can reduce the study to spaces that are T_0 and, hence, posets.

At this point, we discuss how to convert a finite topological space into the Hasse diagram of a poset.

Example 2.12. Let $X = \{a, b, c, d\}$ with the following open sets: $\emptyset$, $\{a, b, c, d\}$, $\{c\}$, $\{d\}$, $\{b, d\}$, $\{c, d\}$ and $\{b, c, d\}$, represented by the interiors of the closed curves of Figure 1(a).

Since X is T_0, it is a poset, thus we can talk about the associated Hasse diagram of X. Let's see how it is constructed. We start with the points corresponding to open sets and place them at the bottommost positions. We can compute the open sets U_x for each $x \in X$: $U_a = \{a, b, c, d\}$, $U_b = \{a, b, c, d\} \cap \{b, d\} \cap \{b, c, d\} = \{b, d\}$, $U_c = \{a, b, c, d\} \cap \{c\} \cap \{c, d\} \cap \{b, c, d\} = \{c\}$ and analogously $U_d = \{d\}$. With this, we establish the order relation: $c < a$, but c is not comparable with b or d; $d < a$ and $d < b$, but since $b < a$, we have the chain $d < b < a$.



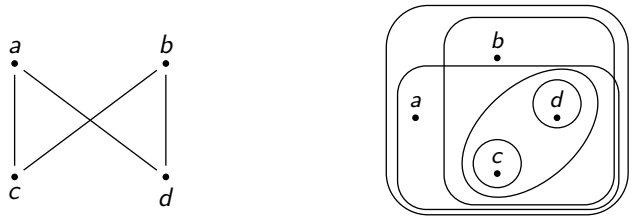
(a) (b)

Figure 1: (a) Open sets of X. (b) Hasse diagram of X.

Now let's see how to obtain a topological space given a the Hasse diagram of a poset.

Example 2.13. Let Y be the poset given by the Hasse diagram seen in Figure 2(a), we want to compute its open sets. Following Definition 2.6, we move through the Hasse diagram starting from the bottom and moving upwards. The sets $\{c\}$ and $\{d\}$ are open.

Now, consider an open set U such that $a \in U$. Since c and d are smaller than a, they must also be in U. Thus, we have the open set $U_a = \{a, c, d\}$. By following a similar process starting from b, we obtain the open set $\{b, c, d\}$. What we have done can be described as "placing our finger" on the point a and descending through all possible edges until reaching the bottom. It is important not to miss any edges, for example, $\{b, d\}$ is not an open set. Therefore, the open sets of X are: $\{c\}$, $\{d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and the unions $\{c, d\}$, $\{a, b, c, d\}$; see Figure 2(b).



(a) (b)

Figure 2: (a) Hasse diagram of space Y. (b) Open sets of Y.

2.2 Bijective correspondence

Finite spaces induce a preorder, and a preorder induces a topology. That is to say, there is a bijective correspondence between finite topological spaces and preorders. Thus, we can talk about the topology of a preorder or the order in a topological space. Table 1 shows a summary of how some properties transfer between finite topological spaces and finite preorders. Some are not explained here, but for further information and details, see the complete final thesis [7].

Finite Topological Space X	Finite Preorder P
U_x	$P_{\leq x}$
$y \in U_x$	$y \leq x$
T_0	Antisymmetric preorder: poset
Diagram of open sets	Hasse diagram
Open set	Down-set
f continuous	f order-preserving
Path, path connected	Fence, order-connected
Homotopy: $f \simeq g$	Fence of maps: $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$

Table 1: Correspondence between finite topological spaces and finite preorders (posets).

3. Stong's Classification Theorem

This section explores how Stong uses homotopy theory to classify finite topological spaces.

3.1 Minimal spaces: the core

We begin by identifying the smallest space that preserves the homotopy properties of a given finite topological space.

Definition 3.1. Let $x, y \in X$ be two points in a finite topological space. We say that x *covers* y if $x > y$ and for all $z \in X$ such that $x > z \geq y$, we have $z = y$. It can also be said that y *is covered* by x.

Definition 3.2. Let X be a finite T_0 topological space. A point $x \in X$ is called a *down beat point* if it covers one and only one element of X. Dually, x is an *up beat point* if it is covered by exactly one element. Points that satisfy either of these properties are referred to as *beat points* of X.

Remark 3.3. In the Hasse diagram, x is a down beat point if it has exactly one lower edge. In the topological space, this is equivalent to saying that the set $\hat{U}_x = U_x \setminus \{x\}$ has a maximum. Similarly, x is an up beat point if it has exactly one upper edge in the Hasse diagram.

We can see in Example 2.12 that b, d and c are up beat points, b is also a down beat point and a is neither of them. There are no beat points in Example 2.13.

Definition 3.4. A finite T_0 topological space X is *minimal* if it has no beat points. The *core* of a finite topological space X is a subspace that is also minimal as a topological space.

Given a finite topological space X, its core can be constructed by removing beat points one at a time. This process preserves the homotopy properties of X because the resulting subspace is a strong deformation retract (see [2, Proposition 1.3.4]).Observe that this minimal subspace always exists. If X has no beat points, it is already minimal, making X its own core, as illustrated by the space in Example 2.13. If beat points are present, they can be removed successively until a minimal space is obtained. For instance, in Example 2.12, we can retract d to b, then b to a, and lastly c to a; therefore, $\{a\}$ is the core of X. As

you may have deduced, the space X is homotopic to a point; therefore, X is indeed contractible, which can be easily observed in the Hasse diagram, rather than in the description of X by its open sets.

Note that minimal is in the sense of not having beat points, not of having a few points. This will be the smallest subspace of a finite topological space that keeps the original homotopy properties. The space of Example 2.12 is not minimal, whereas the one in Example 2.13 is minimal.

Example 3.5. Let X be the finite topological space associated with the Hasse diagram shown in Figure 3(a). We compute its core by removing the beat points. First, since b is an up beat point, we retract it towards a. Then, we retract c towards a because it is an up beat point of $X \setminus \{b\}$. Finally, we retract e towards a because it is an up beat point of $X \setminus \{b, c\}$. The resulting subspace $X \setminus \{b, c, e\}$ is minimal and therefore is the core of X. Note that changing the order of this process leads to the same result, bearing in mind that we can only retract one beat point at a time.

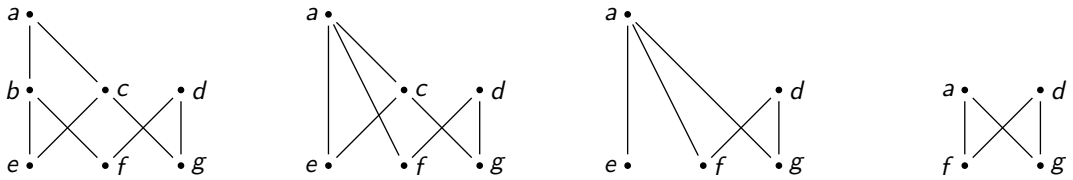


Figure 3: (a) Space X not minimal. (b) $X \setminus \{b\}$. (c) $X \setminus \{b, c\}$. (d) $X \setminus \{b, c, e\}$, the core of X.

3.2 The Theorem

In his work, Stong [6] introduces a matricial approach to classify finite spaces. However, in this paper we will not adopt Stong's method. Instead, the Classification Theorem can be proven using more straightforward propositions, as discussed in [7, Corollary 2.3.10] or [2, Corollary 1.3.7].

Theorem 3.6 (Classification Theorem (Stong))**.** *A homotopy equivalence between minimal finite topological spaces is a homeomorphism. In particular, the core of a finite space is unique up to homeomorphism, and two finite topological spaces are homotopy equivalent if and only if they have homeomorphic cores.*

The crucial point is that with minimality, we can compare spaces by homeomorphism instead of homotopy. Essentially, finite topological spaces are determined up to homeomorphism by their core, there is a *bijection* between posets.

Example 3.7. Consider the following finite T_0 topological spaces X and Y given by Figure 4(a) and (c). They are very similar, but are they homotopic? We already computed the core of X in Example 3.5. For Y, we just retract a, that is a down beat point, towards b and we have the cores shown in Figure 4(c) and (d).

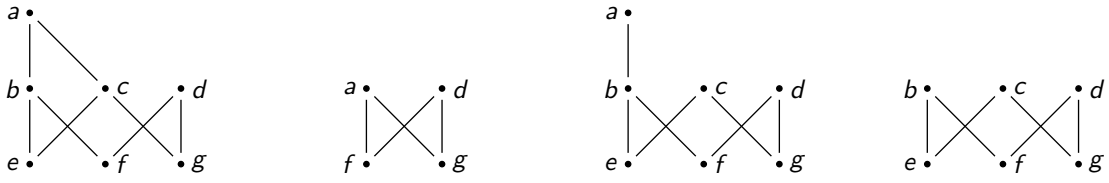


Figure 4: (a) Space X. (b) The core of X. (c) Space Y. (d) The core of Y.

The cores of X and Y are not homeomorphic. Therefore, by Stong's Theorem 3.6, X and Y are not homotopy equivalent. But these two spaces are not visually much different, is there any other way to compare them? The next section is dedicated to relaxing the equivalence criteria to see that they aren't that different.

4. McCord's Theorem

McCord studies finite topological spaces by associating them with abstract combinatorial objects, known as *simplicial complexes*. These complexes can be assigned with a topological representation, called the *geometric realization*, which enables their visualization as geometric structures in Euclidean space, connected coherently, forming geometric shapes such as triangles and tetrahedra. If the reader is unfamiliar with these concepts, they can refer to any introduction to simplicial complexes, such as [5], [2, Appendix] or [7].

4.1 Simplicial complexes

Definition 4.1. A *simplicial complex* $\mathcal{K}$ consists of a set $V_{\mathcal{K}}$, called the set of vertices, and a set $S_{\mathcal{K}}$ of finite nonempty subsets of $V_{\mathcal{K}}$, which is called the set of simplices, satisfying that any subset of $V_{\mathcal{K}}$ of cardinality one is a simplex and any nonempty subset of a simplex is a simplex.

Definition 4.2. The *geometric realization* $|\mathcal{K}|$ of a simplicial complex $\mathcal{K}$ is the set of formal convex combinations $\sum_{v\in\mathcal{K}} \alpha_v v$ such that $\{v|\alpha_v > 0\}$ is a simplex of $\mathcal{K}$.

Definition 4.3. Let X be a T_0 finite topological space. The *simplicial complex associated to* X, or *order complex*, denoted by $\mathcal{K}(X)$, is the simplicial complex whose n-simplices are chains of length n:

$$x_0 < x_1 < \cdots < x_n,$$

where the order relation is given by Proposition 2.4.

Example 4.4. Let X be the finite T_0 topological represented in Figure 5(a). Let's see how to construct the associated simplicial complex $\mathcal{K}(X)$. First, the elements of X are the vertices or 0-simplices. Next, we look at the longest chains. Here we have $\{d < b < a\}$ and $\{e < b < a\}$. Therefore, we have two 2-simplices. Note that they share an edge, $\{b < a\}$.

Next, we go down in dimension. In this case, we need to add an edge from c to d and another from c to e, because the other chains of size 1 are already represented as edges (1-simplices) of the 2-simplices. Finally, we graphically represent the geometric realization of $\mathcal{K}(X)$ in Figure 5(b).

(a) (b)

Figure 5: (a) Space X. (b) Order complex $\mathcal{K}(X)$.

The inverse process can be done by constructing the face poset of $\mathcal{K}$, obtaining a topological space. The face poset is obtained by taking the simplices of a given simplicial complex $\mathcal{K}$ as elements and defining the order relation by inclusion. Note that this would not result in the original space but in a homotopic one.

4.2 The McCord map

Definition 4.5. Given $\mathcal{K}$ and $\mathcal{L}$ two simplicial complexes, a simplicial map $\varphi\colon \mathcal{K} \to \mathcal{L}$ is a vertex map $\varphi'\colon V_{\mathcal{K}} \to V_{\mathcal{L}}$ that sends simplices into simplices.

Note that when defining a map between simplicial complexes, it suffices to specify the map on the vertices, provided that the vertex mapping respects the combinatorial structure of the image complex. This follows from the fact that each simplex in a complex is uniquely determined by its vertices; therefore, once the map is defined on the vertices, it extends naturally and consistently to all simplices.

Definition 4.6. Let X and Y be finite T_0 topological spaces, and $f\colon X \to Y$ a continuous map. Then, the *associated simplicial map* $\mathcal{K}(f)\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\mathcal{K}(f)(x) = f(x)$.

McCord [4, Theorem 6] proved that given a continuous map that is locally a weak homotopy equivalence over a basis-like open cover, then it is globally so. We will use this theorem, the previous definition and the minimal basis to show that the two spaces we have seen before are weak homotopy equivalents.

Example 4.7. As previously discussed in Example 3.5, no homotopy equivalence exists between X and Y because their cores are not homeomorphic. But using the following map, we can prove that there is a weak homotopy equivalence between X and Y. Consider X' and Y' the cores of X and Y, respectively. The map $f\colon Y' \to X'$ is given by $f(a_1) = f(a_2) = f(a_3) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$; see Figure 6.

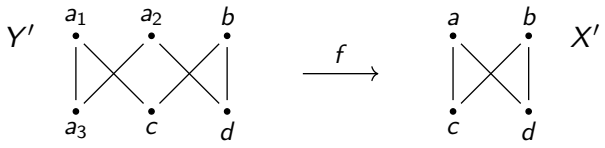


Figure 6: Map f.

It is shown that f is order-preserving, and therefore continuous. Then, by the fact that preimages of minimal open sets are contractible [7, Corollary 2.2.12] and [4, Theorem 6], we obtain that f is a weak homotopy equivalence.

Every homotopy equivalence induces an isomorphism on the homotopy groups, but two spaces with isomorphic homotopy groups may not be homotopy equivalent. This correspondence is called a *weak homotopy equivalence*. This means that spaces X and Y have isomorphic homotopy groups, and with this relaxed criteria, we can finally say that they are equivalent. In fact, they both are finite models of the sphere S^1.

Observe that, given a finite T_0 topological space X and its geometric realization $|\mathcal{K}(X)|$, any point $\alpha \in |\mathcal{K}(X)|$ can be expressed, by construction, in terms of coordinates over a chain $x_1 < x_2 < \cdots < x_n$ in X in the form $\alpha = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i > 0$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} \lambda_i = 1$. The *support* of α is precisely this chain: $\operatorname{supp}(\alpha) = \{x_1, x_2, \dots, x_n\}$.

Definition 4.8. Let X be a finite T_0 topological space and $\alpha \in |\mathcal{K}(X)|$ a point in the geometric realization of the simplicial complex associated with X such that $\text{supp}(\alpha) = \{x_1, x_2, \dots, x_n\} \subseteq X$. The *McCord map* is the map $\mu_X \colon |\mathcal{K}(X)| \to X$ defined by

$$\mu_X(\alpha) = \min(\text{supp}(\alpha)) = x_1.$$

Example 4.9. We start from Example 4.4, where we computed the simplicial complex associated with a topological space X. A point $\alpha \in |\mathcal{K}(X)|$ in the interior of the triangle abe can be written as $\alpha = \lambda_1 e + \lambda_2 b + \lambda_3 a$ with $\sum_{i=1}^{3} \lambda_i = 1$. Then, $\mu_X(\alpha) = \min(\{e, b, a\}) = e$, and therefore every point in the interior of this triangle will map to e, represented in green in Figure 7. Similarly, points in the interior of adb will map to d, represented in red.

Now consider a point on the interior of the edge ae. Since the support is $\{e, a\}$, the minimum is e. More generally, the smallest vertex, concerning the poset order, of all the ones contained in the simplex, will "absorb" the points in the interior of the simplex. Figure 7 shows the McCord map, where colours represent the preimages of the vertices.

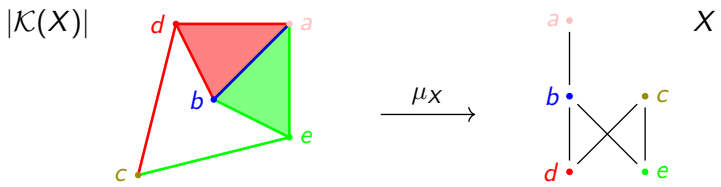


Figure 7: McCord map coloured.

From here, the objective is to conclude this section by establishing that for any given polyhedron, there exists a corresponding finite topological space, and conversely, for any finite topological space, there exists a polyhedron that models it. First, we prove that this correspondence is a weak homotopy, followed by its application in proving McCord's Theorem. The proofs provided by Barmak [2, Theorem 1.4.6] and McCord [4, Theorem 1] are somewhat intricate or may lack comprehensive details. For a clearer comprehension of this demonstration, you can refer to [7, Theorem 3.1.8].

Theorem 4.10 (McCord [4, Theorem 1])**.**

1. *For every finite topological space X, there exists a finite simplicial complex $\mathcal{K}$ and a weak homotopy equivalence $f \colon |\mathcal{K}| \to X$.*
2. *For every finite simplicial complex $\mathcal{K}$, there exists a finite topological space X and a weak homotopy equivalence $f \colon |\mathcal{K}| \to X$.*

This theorem states that every finite topological space has an associated simplicial complex that preserves properties up to weak homotopy equivalence and vice versa, that every simplicial complex has a finite topological space that is weakly homotopy equivalent. Both finite topological spaces and simplicial complexes have a strong combinatorial structure, which is convenient to work with. This is useful because it allows us to study non-finite topological spaces algorithmically through finite ones, by using triangulations of spaces such as the sphere.

Recall that a triangulation is a homeomorphism between a topological space and a simplicial complex. For example, a triangle is homeomorphic to S^1, or a hollow tetrahedron is homeomorphic to S^2, and these are simplicial complexes.

4.3 Finite models of non-finite spaces

We conclude this work by giving some examples of use to model the compact connected surfaces. Due to space constraints, we will not see the 2-torus in this paper; however, it can be found in [7, pp. 32–34]. Beginning with the sphere S^2, consider the following triangulation h given by a hollow tetrahedron; see Figure 8.

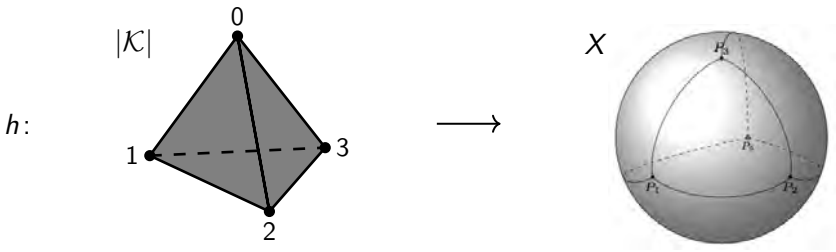


Figure 8: Triangulation of a sphere. Sphere is extracted from [3].

Each vertex is labelled with a number, and each simplex is labelled with the numbers of its vertices. We then construct the face poset of $\mathcal{K}$ and obtain the Hasse diagram of the associated poset; see Figure 9. Note that this gives us a *minimal finite* topological space that models the non-finite space S^2.

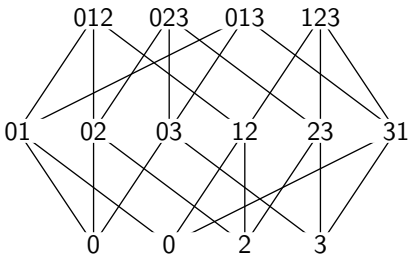


Figure 9: Finite model of S^2.

Also, note that a triangulation with additional vertices would result in a larger finite topological space, although it would also be minimal. This highlights the fact that minimal refers to not having beat points, not to having a few points. Despite being weakly homotopy equivalent, these spaces wouldn't be homotopy equivalent, since they wouldn't be homeomorphic. This underlines why we need to relax the equivalence criteria because we know that they both model the same surface, therefore they must be equivalent in some context.

The next example is the projective plane RP^2. Consider the triangulation illustrated in Figure 10. The Hasse diagram of its associated face poset can be seen in Figure 11.

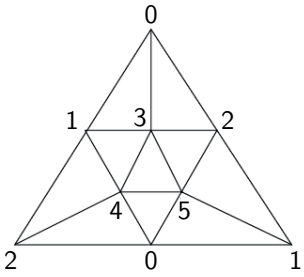


Figure 10: Triangulation of RP^2.

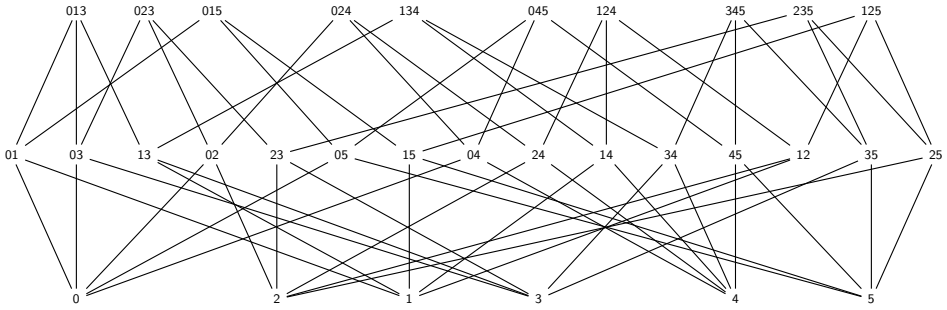


Figure 11: Finite model of RP^2.

The Classification Theorem for compact connected surfaces states that every compact connected surface can be described in terms of spheres, tori, projective planes, and connected sums of these. As a final point, we will provide an overview of how to compute the connected sum, allowing us to model all compact connected surfaces by using the previously provided models and incorporating this procedure.

Consider two spheres S^2 triangulated as in Figure 8. The connected sum consists of identifying vertices and corresponing edges, and eliminating the faces comprised within these vertices; see Figure 12. We proceed as follows: identify vertex 0 of the first sphere with vertex 0 of the second (denoted as $0'$), vertex 2 of the first with vertex 1 of the second (denoted as 4), and vertex 3 of the first with vertex 2 of the second (denoted as $2'$), while keeping vertices 1 of the first and 3 of the second unchanged, identify the corresponding edges, and finally remove the interior triangle formed by $0'$, $2'$ and 4.

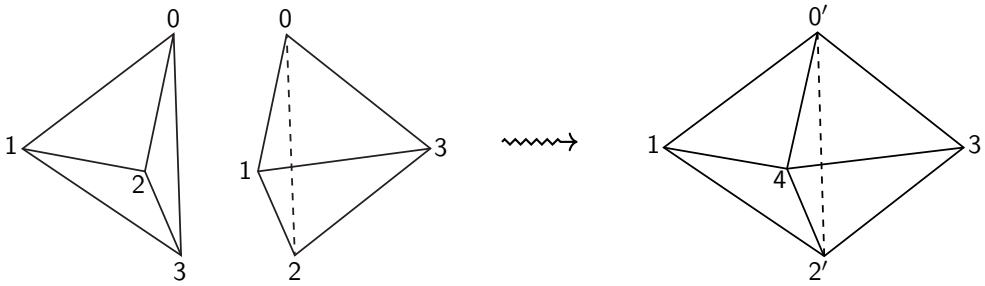


Figure 12: Diagram of the connected sum of two spheres S^2.

Given two associated posets of the spheres, as in Figure 9, we construct the associated poset of the connected sum by following the mentioned procedure: identifying corresponding vertices and omitting the identified triangle; see Figure 13. This approach avoids the need to calculate the poset for the resulting connected sum of spheres.

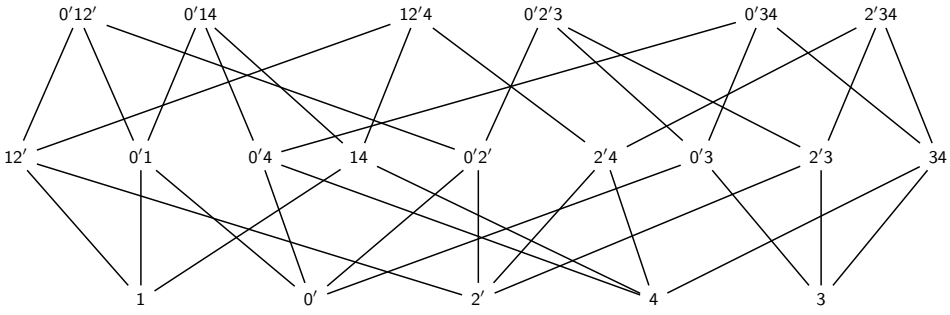


Figure 13: Finite model of a connected sum of two spheres S^2.

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Bernstein–Sato theory for linearly square-free polynomials in positive characteristic

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Resum *(CAT)*
La teoria de Bernstein–Sato ha esdevingut recentment un tema central a l'àlgebra commutativa i la geometria algebraica, atès que constitueix una poderosa eina per a classificar i quantificar singularitats en varietats algebraiques. En particular, ha sorgit un gran interès per estendre la teoria a anells de característica positiva. En aquest article, considerem una classe de polinomis, que denominem polinomis linealment lliures de quadrats, i investiguem els seus invariants associats en el context de la teoria de Bernstein–Sato.

Abstract *(ENG)*
Bernstein–Sato theory has recently emerged as a central topic in commutative algebra and algebraic geometry, as it constitutes a powerful tool in classifying and quantifying singularities of algebraic varieties. Notably, there has been a surge of interest in extending this theory to the positive characteristic setting. In this work, we consider a class of polynomials, which we call linearly square-free polynomials, and investigate their associated invariants within the context of Bernstein–Sato theory.

Keywords: *Bernstein–Sato theory, positive characteristic, test ideals, F-jumping numbers, Bernstein–Sato roots, linearly square-free polynomials.*
MSC (2020): *Primary 13A35, 14F10. Secondary 14B05, 13N10.*

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1. Introduction

A central challenge driving the development of algebraic geometry is the classification of algebraic varieties, which includes the classification of the singularities of these varieties. One approach at tackling this problem is to characterize singularities by attaching algebraic invariants.

A rich family of such invariants falls under the umbrella of the so-called Bernstein–Sato theory, whose roots lie in the foundational works of Bernstein [2] and Sato [23]. We briefly outline their discovery. Denote by $\mathcal{D}_{R|\mathbb{C}}$ the ring of $\mathbb{C}$-linear differential operators on the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ and let f be a nonzero polynomial. Then there exist a nonzero differential operator $\delta(s) \in \mathcal{D}_{R|\mathbb{C}}[s]$, and a non-constant monic polynomial $b_f(s) \in \mathbb{C}[s]$ satisfying the functional equation

$$\delta(s) \cdot f^{s+1} = b_f(s)\, f^s, \quad \text{for } s \in \mathbb{Z}_{\geq 0}.$$

The polynomial $b_f(s)$ is the Bernstein–Sato polynomial of f.

The Bernstein–Sato polynomial has been the focus of extensive research since it encodes the behavior of the singularities of the hypersurface defined by f in $\mathbb{C}^n$. To showcase this, suppose f vanishes at $0 \in \mathbb{C}^n$. A well-known invariant from complex analysis is the log-canonical threshold of f at the origin, defined as

$$\operatorname{lct}(f) = \sup\left\{\lambda \in \mathbb{R}_{>0} \;\middle|\; \int_U \frac{1}{|f|^{2\lambda}} < \infty \text{ for some neighborhood } U \text{ of the origin}\right\}.$$

The log-canonical threshold is a rational number in the interval $(0, 1]$. The more singular f is, the smaller the log-canonical threshold will be. Kollár proved that the log-canonical threshold of f is the smallest root of $b_f(-s)$ [13]. It is known that the roots of $b_f(s)$ are rational and negative due to Malgrange and Kashiwara [16, 12]. A number of invariants have originated around the Bernstein–Sato polynomial over the years. Of special interest in birational geometry are multiplier ideals and jumping numbers (for instance, see [15]).

In positive characteristic, Bernstein–Sato theory has a more recent development. Let us make an overview of one of the main objects of study, namely, the test ideals. These were introduced by Hochster and Huneke as an auxiliary tool in the context of tight closure theory [11], and afterwards related to the multiplier ideals by Hara and Yoshida [8]. Blickle, Mustaţă and Smith gave an alternative but equivalent definition of test ideals in [6], on which we base our study.

To fix ideas, let R be a regular ring of characteristic $p > 0$ and f a nonzero element. The test ideals (cf. Definition 2.10) are a family $\{\tau(f^\lambda)\}_{\lambda \in \mathbb{R}_{\geq 0}}$ of ideals of R indexed by the real numbers. For $\lambda \leq \mu$, these satisfy $\tau(f^\lambda) \supseteq \tau(f^\mu)$, hence one obtains a descending chain of ideals in R. One can show that for a fixed $\lambda > 0$, there exists $\varepsilon > 0$ such that $\tau(f^\lambda) = \tau(f^\mu)$ for all $\mu \in [\lambda, \lambda + \varepsilon)$, i.e. the family is right semi-continuous. On the contrary, there exist certain $\lambda > 0$ such that $\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda)$ for any $\varepsilon > 0$, that is, the chain of test ideals "jumps". These jumping spots are named F-jumping numbers (cf. Definition 2.14), and the smallest among them is the F-pure threshold, as introduced in [24]. Under finiteness hypotheses, F-jumping numbers are known to be discrete and rational (see Theorem 3.1 of [6]). Needless to say, these notions have been extended to non-principal ideals.

As the terminology suggests, the test ideals, F-jumping numbers and F-pure thresholds serve as characteristic $p > 0$ analogues to the multiplier ideals, jumping numbers and log-canonical thresholds, respectively. Remarkably, there is a deep and intricate relationship between these two theories. For instance, one can

recover the log-canonical threshold from the F-pure threshold by letting $p \to \infty$ (see Theorem 3.4 in [19]). It is also known in several cases that the reduction modulo p of a multiplier ideal produces the corresponding test ideal [20].

The F-pure threshold has been computed in a handful of cases. It is known, for instance, in the case of elliptic curves, Calabi–Yau hypersurfaces, diagonal hypersurfaces and determinantal ideals, to name a few [3, 4, 10, 17]. Among the few situations where test ideals have been fully characterized, there is the case of determinantal ideals of maximal minors [9].

In general, finding F-jumping numbers and test ideals is a challenging problem, even in smooth ambient spaces such as polynomial rings and with the aid of computational tools. To some extent, the aforementioned known results rely on the favorable arithmetic and combinatorial properties of the objects involved. Without these properties, very little can be said about F-invariants.

Our goal in this article is to compute the F-jumping numbers and test ideals for a new class of polynomials, which we refer to as linearly square-free polynomials. These are polynomials whose monomials are all square-free, meaning they are not divisible by any square of an indeterminate. In the process, we also compute several other F-invariants useful for the theory, namely, the ν-invariants, Frobenius roots, and Bernstein–Sato roots, which we will introduce in due course. Finally, we relate these computations to the log-canonical threshold of linearly square-free polynomials in characteristic zero. This work originated from the study of F-invariants for determinants of generic matrices of indeterminates in characteristic $p > 0$. Subsequently, it was realized that the same ideas applied to linearly square-free polynomials.

Throughout, all rings considered will be commutative with unit.

2. Background

2.1 Frobenius powers and Frobenius roots

Let R be a ring of characteristic $p > 0$. We denote by $F\colon R \to R$, $f \mapsto f^p$ the Frobenius or p-th power map. This is a ring endomorphism of R. For an integer $e \geq 0$, we let $F^e\colon R \to R$, $f \mapsto f^{p^e}$ be the e-th iterate of the Frobenius.

Definition 2.1. For an integer $e \geq 0$, the e-th Frobenius power of an ideal $I \subseteq R$ is

$$I^{[p^e]} = F^e(I)R = (f^{p^e} \mid f \in I).$$

This is an ideal of R. In the case that I be generated by $f_1, \dots, f_n$, one has

$$I^{[p^e]} = (f_1^{p^e}, \dots, f_n^{p^e}).$$

Remark 2.2. When I is a principal ideal of R, say $I = (f)$, Frobenius powers and the usual powers coincide,

$$(f)^{[p^e]} = (f)^{p^e}.$$

A sort of converse operation to Frobenius powers are Frobenius roots. For principal ideals, Frobenius roots were first introduced in [1] by Àlvarez-Montaner, Blickle and Lyubeznik, in order to study generators of modules over rings of differential operators in positive characteristic. Afterwards, Frobenius roots were generalized to the non-principal case by Blickle, Mustaţă and Smith in [6].

Definition 2.3. For an integer $e \geq 0$, the e-th Frobenius root of an ideal $I \subseteq R$ is the smallest ideal $J \subseteq R$ in the sense of inclusion such that

$$I \subseteq J^{[p^e]}.$$

We denote the e-th Frobenius root of the ideal I by $I^{[1/p^e]}$. For $e = 0$, we set $I^{[1/p^e]} = I$.

A celebrated theorem of Kunz states that a ring R of characteristic $p > 0$ is regular if and only if the Frobenius $F\colon R \to R$ is a flat map [14]. Under the assumption of regularity, one can show that Frobenius roots are well-defined. See, for instance, Lemma 2.3 of [6].

Remark 2.4. Let I_1, I_2 be ideals of R such that $I_1 \subseteq I_2$. Then one has

$$I_1 \subseteq I_2 \subseteq (I_2^{[1/p^e]})^{[p^e]}.$$

Because $I_1^{[1/p^e]}$ is the smallest ideal with $I_1 \subseteq (I_1^{[1/p^e]})^{[p^e]}$, it follows that

$$I_1^{[1/p^e]} \subseteq I_2^{[1/p^e]},$$

hence Frobenius roots preserve ideal containments.

Remark 2.5. Let I, J be ideals of R and $e \geq 0$ an integer. Then

$$I \cdot J^{[1/p^e]} \subseteq (I^{[p^e]} J)^{[1/p^e]}.$$

Proposition 2.6 ([21, Lemma 2.3])**.** *Let I, J be ideals of R and $e \geq 0$ an integer. One has that $I^{[1/p^e]} \subseteq J$ if and only if $I \subseteq J^{[p^e]}$.*

We next describe a nice characterization of Frobenius roots in terms of generators, which will prove to be computationally useful. To this end, we endow R with an exotic R-module structure.

Definition 2.7. For an integer $e \geq 0$, define the R-module $F_*^e R$ as follows. Its elements are denoted by $F_*^e f$, where f is in R. As an abelian group, $F_*^e R$ is isomorphic to R, so addition is defined by

$$F_*^e f + F_*^e g = F_*^e (f + g), \quad \text{for } F_*^e f, F_*^e g \in F_*^e R.$$

The action of R on $F_*^e R$ is defined by restricting scalars along the e-th iterate F^e of the Frobenius, that is,

$$r \cdot F_*^e f = F_*^e (r^{p^e} f), \quad \text{for } r \in R,\ F_*^e f \in F_*^e R.$$

Definition 2.8. A Noetherian ring R of characteristic $p > 0$ is an F-finite ring if $F_*^e R$ is a finitely generated R-module for some $e \geq 1$ (equivalently, all $e \geq 1$).

Proposition 2.9 ([1, Section 3], [6, Proposition 2.5])**.** *Suppose that $F_*^e R$ is a free R-module with basis $\varepsilon_1, \ldots, \varepsilon_n$. Let I be an ideal of R generated by $f_1, \ldots, f_m$. For a generator f_i, $i = 1, \ldots, m$, write*

$$F_*^e f_i = g_{i,1} F_*^e \varepsilon_1 + \cdots + g_{i,n} F_*^e \varepsilon_n, \quad \text{where } g_{i,1}, \ldots, g_{i,n} \in R.$$

Then the e-th Frobenius root of I is

$$I^{[1/p^e]} = (g_{i,j} \mid i = 1, \ldots, m,\ j = 1, \ldots, n).$$

2.2 Test ideals and ν-invariants

From now on, let R be a regular F-finite ring of characteristic $p > 0$. For a real number $x \in \mathbb{R}$, let $\lceil x \rceil \in \mathbb{Z}$ denote the round-up of x, i.e. the least integer greater or equal than x.

As mentioned earlier, the test ideals are the characteristic $p > 0$ analogues of the multiplier ideals. We adopt as a definition for the test ideal the characterization given in [6]:

Definition 2.10 ([6, Definition 2.9])**.** The test ideal of an ideal $I \subseteq R$ with exponent $\lambda \in \mathbb{R}_{\geq 0}$ is

$$\tau(I^\lambda) = \bigcup_{e=0}^{\infty} (I^{\lceil \lambda p^e \rceil})^{[1/p^e]}.$$

Remark 2.11. It can be shown that the ideals on the right-hand side give an ascending chain in R,

$$(I^{\lceil \lambda p \rceil})^{[1/p]} \subseteq (I^{\lceil \lambda p^2 \rceil})^{[1/p^2]} \subseteq \cdots \subseteq (I^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (I^{\lceil \lambda p^{e+1} \rceil})^{[1/p^{e+1}]} \subseteq \cdots$$

(see Lemma 2.8 in [6]). Since R is a Noetherian ring, the chain eventually stabilizes:

$$\tau(I^\lambda) = (I^{\lceil \lambda p^e \rceil})^{[1/p^e]}, \quad \text{for some } e \gg 0.$$

Remark 2.12. Let $0 \leq \lambda \leq \mu$ be real numbers. Because $\lceil \lambda p^e \rceil \leq \lceil \mu p^e \rceil$, one has that

$$I^{\lceil \lambda p^e \rceil} \supseteq I^{\lceil \mu p^e \rceil}, \quad \text{for every } e \geq 1.$$

On the other hand, Remark 2.4 shows that Frobenius roots preserve inclusions, therefore

$$\tau(I^\lambda) \supseteq \tau(I^\mu), \quad \text{whenever } \mu \geq \lambda \geq 0.$$

It follows from the remark above that test ideals give a descending chain of ideals of R. More explicitly, given non-negative real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq$, one has that

$$\tau(I^{\lambda_1}) \supseteq \tau(I^{\lambda_2}) \supseteq \cdots \supseteq \tau(I^{\lambda_n}) \supseteq \cdots.$$

Such chain of ideals can "jump", i.e. the containments between test ideals may be strict. The results below encode this behavior:

Theorem 2.13 ([19, Remark 2.12], [6, Corollary 2.16, Theorem 3.1])**.** *Let I be an ideal of R.*

(i) For each $\lambda \geq 0$, there exists $\varepsilon > 0$ such that $\tau(I^\lambda) = \tau(I^{\lambda+\varepsilon})$. In particular, there exists $\lambda > 0$ small enough such that $\tau(I^\lambda) = R$.

(ii) There exist real numbers $\lambda > 0$ such that $\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda)$ for all $\varepsilon > 0$.

Definition 2.14 ([24, Definition 2.1], [19], [6, Definition 2.17])**.** Let I be an ideal of R. A real number $\lambda > 0$ is an F-jumping number of I if

$$\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda), \quad \text{for every } \varepsilon > 0.$$

The smallest F-jumping number is called the F-pure threshold of I, and denoted by $\mathrm{fpt}(I)$, namely

$$\mathrm{fpt}(I) = \sup\{\lambda > 0 \mid \tau(I^\lambda) = R\}.$$

F-jumping numbers were introduced under the name F-thresholds in [19], as an invariant to study the jumping coefficients of the test ideals of Hara and Yoshida [8]. Afterwards, it was shown that the sets of F-thresholds and F-jumping numbers are equal (see [6, Corollary 2.30]). On another note, one has the following result relating the log-canonical threshold and the F-pure threshold:

Theorem 2.15 ([19, Theorem 3.4]). *Let f be a polynomial with integer coefficients in $\mathbb{C}[x_1, \dots, x_n]$. For a prime number $p > 0$, let f_p denote the reduction modulo p of f in $\mathbb{F}_p[x_1, \dots, x_n]$. Then*

$$\lim_{p\to\infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f).$$

Another object closely related to the F-jumping numbers are the ν-invariants:

Definition 2.16 ([19]). Let I, J be ideals of R such that $I \subseteq \operatorname{rad} J$, where $\operatorname{rad} J$ denotes the radical of J. Fix an integer $e \geq 0$. The ν-invariant of level e of I with respect to J is

$$\nu_I^J(p^e) = \max\{r \in \mathbb{Z} \mid I^r \not\subseteq J^{[p^e]}\}.$$

Because $I \subseteq \operatorname{rad} J$, this integer exists and is finite. The set $\nu_I^\bullet(p^e)$ of ν-invariants of level e of I is the set of integers of the form $\nu_I^J(p^e)$ obtained as J ranges over the ideals containing I in its radical:

$$\nu_I^\bullet(p^e) = \{\nu_I^J(p^e) \mid J \subseteq R \text{ such that } I \subseteq \operatorname{rad} J\}.$$

Remark 2.17. In view of Proposition 2.6, $r \geq 0$ is the ν-invariant $\nu_I^J(p^e)$ if and only if $(I^r)^{[1/p^e]} \not\subseteq J$.

The ν-invariants were introduced precisely to study F-thresholds. In fact, the F-threshold $c^J(I)$ of I with respect to J was defined in [19] as

$$c^J(I) = \lim_{e\to\infty} \frac{\nu_I^J(p^e)}{p^e}.$$

Since F-thresholds and F-jumping numbers coincide when R is a regular ring, the ν-invariants are a powerful tool for shedding light on test ideals.

In computing the ν-invariants of an ideal I, it is not evident how to choose an ideal J that contains I in its radical. Instead, however, one can inspect the chain of ideals

$$\dots \subseteq (I^{r+1})^{[1/p^e]} \subseteq (I^r)^{[1/p^e]} \subseteq \dots \subseteq (I^2)^{[1/p^e]} \subseteq I^{[1/p^e]} \subseteq R.$$

In some cases, the containments are, in fact, equalities. When they are not, the chain of ideals "jumps". The next proposition, together with Remark 2.17, shows that the spot where the chain jumps is a ν-invariant.

Proposition 2.18 ([21, Proposition 4.2]). *The set of ν-invariants of level $e \geq 0$ of an ideal I is*

$$\nu_I^\bullet(p^e) = \{r \geq 0 \mid (I^{r+1})^{[1/p^e]} \neq (I^r)^{[1/p^e]}\}.$$

2.3 Bernstein–Sato roots

The last algebraic invariants relevant to our discussion are the Bernstein–Sato roots. These are characteristic $p > 0$ analogues to the roots of the Bernstein–Sato polynomial in characteristic zero, a concept

that originated from Mustaţă's work [18]. Mustaţă initiated the extension of Bernstein–Sato polynomials to positive characteristic, an effort further advanced by Bitoun [5]. Due to the intricate nature of constructing Bernstein–Sato roots, we will instead use the more straightforward characterization in terms of ν-invariants, as provided by Quinlan-Gallego [21]. Before delving into this topic, we will briefly discuss p-adic limits and integers.

The p-adic valuation on $\mathbb{Z}$ is the map $v_p\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by $v_p(0) = \infty$ and

$$v_p(n) = \max\{k \geq 0 \mid p^k \text{ divides } n\}, \quad \text{for } n \neq 0,$$

which naturally extends to a valuation $v_p\colon \mathbb{Q} \to \mathbb{Z}_{\geq 0}$ by letting

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b).$$

This induces the p-adic norm $|\cdot|_p\colon \mathbb{Q} \to \mathbb{R}$, $|x|_p = p^{-v_p(x)}$, and in turn the p-adic metric $d_p\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$, $d_p(x, y) = p^{-v_p(x-y)}$. In this setting, the ring $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic metric. The ring $\mathbb{Z}_p$ of p-adic integers is the subring of $\mathbb{Q}_p$ given by

$$\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p \mid |\alpha|_p \leq 1\}.$$

Because $v_p(n) \geq 0$ for every $n \in \mathbb{Z}$, one has $|n|_p \leq 1$, therefore $\mathbb{Z}$ is contained in $\mathbb{Z}_p$. From the definition, one also sees that $\mathbb{Q}$ is contained in $\mathbb{Q}_p$. A sequence $(x_n)_{n=0}^{\infty} \subseteq \mathbb{Q}$ has p-adic limit $\alpha \in \mathbb{Q}_p$ if $x_n \to \alpha$ in the p-adic metric. For more on p-adic numbers, we refer the interested reader to Section 7 in [21].

With this in mind, Bernstein–Sato roots are defined as follows:

Definition 2.19 ([21, Proposition 6.13], [22, Theorem IV.17])**.** Let I be an ideal of R. A p-adic integer $\alpha \in \mathbb{Z}_p$ is a Bernstein–Sato root of I if there exists a sequence $(\nu_e)_{e=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ of ν-invariants of I, $\nu_e \in \nu_I^{\bullet}(p^e)$, whose p-adic limit is α.

3. Linearly square-free polynomials

In this section we prove our main results, namely, the computation of Bernstein–Sato theory invariants for linearly square-free polynomials in characteristic $p > 0$.

Definition 3.1. Let $R = B[x_1, \ldots, x_n]$ be a polynomial ring over a commutative ring B. We say that a polynomial in R is a linearly square-free polynomial if all its monomials are square-free.

Example 3.2. Let $R = B[x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}]$ be a polynomial ring in n^2 indeterminates. The indeterminates may be assembled in an $n \times n$ generic matrix of indeterminates $X = (x_{ij})$. Then the determinant of X,

$$\det X = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

is a linearly square-free polynomial.

Example 3.3. Let $X = (x_{ij})$ be a $2n \times 2n$ skew-symmetric matrix of indeterminates, that is, $x_{ij} = -x_{ji}$ for $1 \leq i, j \leq 2n$. The Pfaffian of X is the polynomial

$$\operatorname{Pf} X = \frac{1}{2^n n!} \sum_{\sigma \in \operatorname{Sym}(2n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(2i-1)\sigma(2i)}.$$

It can be shown that the Pfaffian satisfies $(\operatorname{Pf} X)^2 = \det X$. Since no indeterminate appears twice in the same monomial, the Pfaffian is linearly square-free.

Example 3.4. Let K be a field and $W \subseteq K^E$ be a realization of a matroid M, where E is a finite set that forms a basis of K^E. Then the configuration polynomial of W is linearly square-free (see [7]). These polynomials have applications in physics.

The proposition below is a well-known fact that shows that $F_*^e R$ has a particularly nice structure provided R is a polynomial ring over a perfect field of characteristic $p > 0$. Recall that a field K of characteristic $p > 0$ is perfect if the Frobenius $F\colon K \to K$ is an automorphism of K. This is tantamount to every element of K having a p^e-th root in K.

Proposition 3.5. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field K of characteristic $\operatorname{Char}(K) = p > 0$. For each integer $e \geq 0$, one has that*

$$F_*^e R \simeq \bigoplus_{0 \leq i_1, \ldots, i_n < p^e} R\, F_*^e x_1^{i_1} \cdots x_n^{i_n}.$$

In consequence, the set $\{F_*^e x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \ldots, i_n < p^e\}$ is a basis for $F_*^e R$. We refer to this as the standard basis of $F_*^e R$.

We start by computing the Frobenius roots and the ν-invariants of linearly square-free polynomials. This will lay the groundwork for further results. For the following lemma, it will be convenient to use multi-index notation. If $B[x_1, \ldots, x_n]$ is a polynomial ring in n variables, and $a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ is an n-tuple of non-negative integers, we let

$$x^a = x_1^{a_1} \cdots x_n^{a_n}.$$

Lemma 3.6. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. Let f be a linearly square-free polynomial. Fix an integer $e \geq 0$. Then for all integers $0 \leq r < p^e$, $F_*^e f^r$ is a nonzero K-linear combination of elements in the standard basis of $F_*^e R$.*

Proof. Because f is linearly square-free, one has

$$f = \sum_{i=1}^{m} \alpha_i x^{a_i}, \quad \text{where } \alpha_i \in K,\ a_i = (a_{i1}, \ldots, a_{in}) \in \{0,1\}^n,$$

for some integer $m \geq 1$, therefore

$$f^r = \sum_{k_1 + \cdots + k_m = r} \binom{r}{k_1, \ldots, k_m} \prod_{i=1}^{m} \alpha_i^{k_i} x^{k_i a_i}.$$

The monomials in the expression above have the form

$$\prod_{i=1}^{m} x^{k_i a_i} = x^{\sum_{i=1}^m k_i a_i} = x_1^{\sum_{i=1}^m k_i a_{i1}} \cdots x_n^{\sum_{i=1}^m k_i a_{in}}.$$

By assumption $0 \leq r < p^e$, hence the indeterminate x_j appears in each monomial with exponent

$$\sum_{i=1}^{m} k_i a_{ij} \leq \sum_{i=1}^{m} k_i = r < p^e.$$

It follows that

$$F_*^e x^{\sum_{i=1}^m k_i a_i}, \quad \text{for } i = 1, \dots, m,$$

is an element in the standard basis of $F_*^e R$. As a result, up to collecting terms, $F_*^e f^r$ reads

$$F_*^e f^r = \sum_{k_1+\cdots+k_m=r} \left(\binom{r}{k_1, \dots, k_m} \prod_{i=1}^{m} \alpha_i^{k_i} \right)^{1/p^e} F_*^e x^{\sum_{i=1}^m k_i a_i},$$

which proves that the coefficients are in K. Because $f^r \neq 0$ and $F_*^e R$ is a free R-module, some coefficient is nonzero. □

Theorem 3.7. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. Let f be a linearly square-free polynomial. Fix an integer $e \geq 0$. Then:*

(i) For all integers $s \geq 0$ and $0 \leq r < p^e$,

$$(f^{sp^e+p^e-1})^{[1/p^e]} = (f^{sp^e+p^e-2})^{[1/p^e]} = \cdots = (f^{sp^e+1})^{[1/p^e]} = (f^{sp^e})^{[1/p^e]} = (f)^s.$$

(ii) The ν-invariants of f of level e are $\nu_f^\bullet(p^e) = \{(s+1)p^e - 1 \mid s \in \mathbb{Z}_{\geq 0}\}$.

(iii) If $s \geq 0$ is an integer and $J = (f)^{s+1}$, then $\nu_f^J(p^e) = (s+1)p^e - 1$.

Proof. (i) For a fixed integer $s \geq 0$, Frobenius roots give an ascending chain

$$(f^{sp^e+p^e-1})^{[1/p^e]} \subseteq (f^{sp^e+p^e-2})^{[1/p^e]} \subseteq \cdots \subseteq (f^{sp^e+1})^{[1/p^e]} \subseteq (f^{sp^e})^{[1/p^e]}.$$

In the case $s = 0$, Lemma 3.6 shows that $F_*^e f^{p^e-1}$ is a nonzero K-linear combination of elements in the standard basis of $F_*^e R$. It follows from Proposition 2.9 that the Frobenius root $(f^{p^e-1})^{[1/p^e]}$ is generated by units of R, therefore $(f^{p^e-1})^{[1/p^e]} = R$. Now suppose that $s \geq 1$. In view of the ascending chain above, to prove equality it suffices to verify that

$$(f)^s \subseteq (f^{sp^e+p^e-1})^{[1/p^e]} \quad \text{and} \quad (f^{sp^e})^{[1/p^e]} \subseteq (f)^s.$$

On the one hand, by Remark 2.5,

$$(f)^s = (f)^s (f^{p^e-1})^{[1/p^e]} = (f^{s[p^e]} f^{p^e-1})^{[1/p^e]} = (f^{sp^e+p^e-1})^{[1/p^e]}.$$

On the other hand, by Proposition 2.6, the containment $(f^{sp^e})^{[1/p^e]} \subseteq (f)^s$ is equivalent to $(f)^{sp^e} \subseteq (f)^{s[p^e]} = (f)^{sp^e}$.

(ii) Part (i) shows that for each integer $s \geq 0$,

$$(f)^{s+1} = (f^{(s+1)p^e})^{[1/p^e]} \subsetneq (f^{(s+1)p^e-1})^{[1/p^e]} = (f)^s.$$

Then by Proposition 2.18, the ν-invariants of f of level $e \geq 0$ are of the form $(s+1)p^e - 1$ for $s \in \mathbb{Z}_{\geq 0}$.

(iii) It follows at once from Definition 2.16 and part (ii). □

Lemma 3.8. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$ and f be a linearly square-free polynomial. Let $\lambda \geq 0$ be a real number and $e \geq 0$ an integer. Then*

$$(f^{\lceil \lambda p^e \rceil})^{[1/p^e]} = \begin{cases} (f)^{\lfloor \lambda \rfloor} & \text{if } \{\lambda\} \leq (p^e-1)/p^e, \\ (f)^{\lfloor \lambda \rfloor + 1} & \text{if } \{\lambda\} > (p^e-1)/p^e, \end{cases}$$

where $\{\lambda\}$ denotes the fractional part of λ.

Proof. Write λ as $\lambda = \lfloor \lambda \rfloor + \{\lambda\}$. If $\{\lambda\} \leq (p^e-1)/p^e$, one has that $\lfloor \lambda \rfloor p^e \leq \lambda p^e \leq \lfloor \lambda \rfloor p^e + p^e - 1$, therefore $\lfloor \lambda \rfloor p^e \leq \lceil \lambda p^e \rceil \leq \lfloor \lambda \rfloor p^e + p^e - 1$. Theorem 3.7 shows

$$(f)^{\lfloor \lambda \rfloor} = (f^{\lfloor \lambda \rfloor p^e + p^e - 1})^{[1/p^e]} \subseteq (f^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (f^{\lfloor \lambda \rfloor p^e})^{[1/p^e]} \subseteq (f)^{\lfloor \lambda \rfloor}.$$

On the other hand, suppose that $\{\lambda\} > (p^e-1)/p^e$. Similarly, one finds $\lfloor \lambda \rfloor p^e + p^e - 1 < \lambda p^e < \lfloor \lambda \rfloor p^e + p^e$, which gives $\lceil \lambda p^e \rceil = \lfloor \lambda \rfloor p^e + p^e$. Again using Theorem 3.7 gives

$$(f^{\lceil \lambda p^e \rceil})^{[1/p^e]} = (f^{(\lfloor \lambda \rfloor + 1)p^e})^{[1/p^e]} = (f)^{\lfloor \lambda \rfloor + 1},$$

thus proving the lemma. □

Theorem 3.9. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. Let f be a linearly square-free polynomial. Then:*

(i) For a real number $\lambda \geq 0$, one has $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor}$.

(ii) The set of F-jumping numbers of f is $\mathrm{FJN}(f) = \mathbb{Z}_{\geq 1}$. In particular, the F-pure threshold of f is 1.

Proof. (i) Since the sequence $((p^e-1)/p^e)_{e=0}^{\infty}$ has limit 1 as $e \to \infty$, there is an integer e_0 satisfying $\{\lambda\} \leq (p^e-1)/p^e$ for all $e \geq e_0$. It follows from Lemma 3.8 that $(f^{\lceil \lambda p^e \rceil})^{[1/p^e]} = (f)^{\lfloor \lambda \rfloor}$ for $e \geq e_0$, therefore $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor}$.

(ii) Fix an integer $n \geq 0$. Then $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor} = (f)^n$ for all real numbers λ with $n \leq \lambda < n+1$. On the other hand, one has $\tau(f^{n+1}) = (f)^{n+1}$. Consequently $n+1$ is an F-jumping number of f, and the assertion follows. □

Corollary 3.10. *Let f be a linearly square-free polynomial with integer coefficients in $\mathbb{C}[x_1, \ldots, x_n]$. The log-canonical threshold of f is $\mathrm{lct}(f) = 1$.*

Proof. Let $p > 0$ be a prime number and f_p be the reduction modulo p of f in $\mathbb{F}_p[x_1, \ldots, x_n]$. If p does not divide all the coefficients of f, then f_p is nonzero and thus linearly square-free, hence $\mathrm{fpt}(f_p) = 1$ by Lemma 3.8. This occurs for all p large enough, so $\mathrm{lct}(f) = 1$ by Theorem 2.15. □

Corollary 3.11. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. The only Bernstein–Sato root of a linearly square-free polynomial f is $\alpha = -\operatorname{fpt}(f) = -1$.*

Proof. Let $(t_d)_{d=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ be a sequence of non-negative integers and define

$$\nu_d := (t_d + 1)p^d - 1, \quad \text{for } d \geq 0.$$

In view of Theorem 3.7, each ν_d is a ν-invariant of f. We thus obtain a sequence $(\nu_d)_{d=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ of ν-invariants with p-adic limit $\nu_d \to \alpha = -1$ as $d \to \infty$. In consequence, $\alpha = -\operatorname{fpt}(f)$ is a Bernstein–Sato root of f. Because any sequence of ν-invariants of f is of this form, it follows that $\alpha = \operatorname{fpt}(f)$ is the only Bernstein–Sato root of f. □

The corollary above allows one to answer the following question.

Question 3.12 ([21, Question 6.16])**.** *Suppose that the F-pure threshold α of an ideal I lies in $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at $\{p^k \mid k \geq 0\}$. Is the largest Bernstein–Sato root of I equal to $-\alpha$?*

The answer is affirmative for linearly square-free polynomials in any characteristic $p > 0$.

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A promenade through singular symplectic geometry

A la memòria del Marc Herault

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Resum *(CAT)*
En aquest article, presentem la geometria simplèctica i de Poisson des de la mecànica hamiltoniana. Després introduïm els algebroides de Lie simplèctics, objectes al mig de la geometria simplèctica i de Poisson. Posteriorment, recordem la noció de reducció simplèctica en presència d'una aplicació moment. Com a aplicació d'aquesta construcció, descrivim els espais de fase de partícules carregades sota la presència de camps de Yang–Mills. Finalment, introduïm un anàleg singular d'aquesta construcció i donem exemples físics.

Abstract *(ENG)*
In this article, we present symplectic and Poisson geometry from the perspective of Hamiltonian mechanics. We then introduce symplectic Lie algebroids, objects which lie between symplectic and Poisson manifolds. Afterwards, we recall the notion of symplectic reduction under the existence of a moment map. As an application of this construction, we describe the phase space of a charged particle interacting with a Yang–Mills field. Finally, we introduce a singular analogue of this construction and provide physical examples.

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1. Introduction

Classical mechanics was inaugurated by the works of Isaac Newton. After his contribution, different approaches to write Newton's equations of motion were developed, commonly with the goal of improving certain aspects of the previous formalism. In Hamiltonian mechanics, the equations of motion are a system of first order ordinary differential equations, known as Hamilton's equations. This feature makes easier discussing qualitative aspects of solutions from the perspective of dynamical systems. Moreover, the dual behavior of symmetries and conserved quantities, originally established by Emmy Noether for Lagrangian systems, becomes transparent in the Hamiltonian formalism.

Symplectic geometry can be regarded as an abstraction of Hamiltonian dynamics for smooth manifolds. Poisson geometry is a further generalization of the symplectic setting, where the relevant structure is the Poisson bracket defining the evolution of observables along the dynamics of the system. As we will see, Poisson structures vastly generalize symplectic structures and, consequently, many results from the symplectic category fail to be transferred to Poisson manifolds. Symplectic Lie algebroids define Poisson structures which, although not arising from a symplectic form, have a very close behavior to them. In physics, these objects allow to describe physical systems with degenerate or constrained dynamics. In mathematics, they have proved to be the adequate language to establish results for a class of Poisson structures.

New discoveries in particle physics during the XX century posed the problem of incorporating the weak and strong forces into mechanics. The satisfactory formulation was proposed by Yang and Mills, and is nowadays known under the name of *gauge theory*.[1] The equations describing the motion of a charged particle under the presence of a Yang–Mills field are a generalization of Lorentz's force equation, and are known as Wong's equations. Sternberg showed how Wong's equations fit into the Hamiltonian formalism of mechanics. Weinstein additionally proved that the phase space constructed by Sternberg could be realized as the reduction of a universal space for particles interacting with Yang–Mills fields.

The goal of this article is to fill the picture introduced in this section. In Section 2 we recall the fundamentals of symplectic and Poisson geometry from the Hamiltonian formalism of mechanics. In Section 3 we introduce Lie algebroids and E-symplectic manifolds as objects between symplectic and Poisson structures. We will additionally give examples of interest where they have been fruitfully applied. In Section 4 we remember the interplay between conserved quantities and symmetries, codified in the moment map of a Hamiltonian action. The presence of symmetries allows for elimination of degrees of freedom, a procedure formalized by the reduction theorem of Marsden and Weinstein. We present Sternberg's and Weinstein's constructions, and show how they have been extended to the setting of E-symplectic manifolds.

2. Symplectic and Poisson geometry

2.1 Symplectic geometry

Symplectic geometry can be considered an abstraction of the Hamiltonian formulation of classical mechanics. In this formalism, the equations of motion in the Euclidean space $\mathbb{R}^{2n}$, described in terms of

[1] In mathematics, gauge theories refer to the study of connections in vector and principal bundles. The name of *Yang–Mills theories* is reserved to the study of solutions to the Yang–Mills equations.

coordinates p_i, q_i, can be recovered from a function $H \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, called the Hamiltonian, following Hamilton's equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1}$$

The previous set of equations can be compactly written using matrix notation as $X_H = \mathbb{J} \cdot \nabla f$, where $\mathbb{J}$ is the standard skew-symmetric matrix. Common choices in physics for the Hamiltonian are energy functions of the form $H = \frac{1}{2m}\sum p_i^2 + V(q)$ for some smooth function $V \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, called the potential of the system.

In many examples, as in systems with constraints, it is better to work directly in the setting of differentiable manifolds. To write the previous set of equations in an abstract manifold, however, we need to choose additional data relating the Hamiltonian vector field X_H and the differential $\mathrm{d}H$. Equation (1) suggests that we should choose a skew-symmetric and non-degenerate tensor $\omega \in \Omega^2(M)$. For many results to hold we have to additionally impose the form ω to be closed. While there is good geometric motivation behind this requirement, we do not have the space to delve into this matter.

Definition 2.1. Let M be a smooth manifold. A non-degenerate, closed two-form $\omega \in \Omega^2(M)$ is called a *symplectic form*. We call any such pair (M, ω) a *symplectic manifold*.

Following the previous analogy between ω and the matrix $\mathbb{J}$, Hamilton's equations of motion (1) should be written in this new language as

$$\iota_{X_H}\omega = -\mathrm{d}H.$$

There is no apparent reason to believe this expression should be related in general to equations (1). It is a theorem of Darboux that this is, indeed, the case. More precisely, we have the following:

Theorem 2.2 (Darboux)**.** *Let (M, ω) be a symplectic manifold. For every point $p \in M$ there exists a chart $\varphi\colon U \subset \mathbb{R}^n \to M$ centered at p with coordinates q_i, p_j such that*

$$\varphi^*\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

This result is very powerful because it shows that symplectic geometry has no local invariants. Consequently, all interesting information in symplectic manifolds has to be of topological/global nature.

2.2 Poisson geometry

Poisson brackets were originally introduced to study the evolution of observables, i.e., smooth functions, along the Hamiltonian dynamics. In more mathematical terms, if we define the Poisson bracket of H and f to be the derivative of f along the flow of X_H, Hamilton's equations (1) directly show

$$\{H, f\} = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}. \tag{2}$$

In the more general setting of symplectic geometry there exists an analogue generalization of the Poisson bracket given by the formula

$$\{f, g\} = \omega(X_f, X_g). \tag{3}$$

Poisson showed that his eponymous bracket (2) is linear in both arguments, skew-symmetric, satisfies Leibniz's rule and Jacobi's identity holds:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0.$$

Even though any bracket arising from (3) satisfies these conditions, there are brackets fulfilling these properties which cannot be defined in this way. A trivial example is the Poisson bracket $\{f,g\}=0$ for all $f,g\in\mathcal{C}^\infty(M)$. The systematic study of these objects is the branch of *Poisson geometry*.

Definition 2.3. A *Poisson bracket* on a smooth manifold M is a bilinear, skew-symmetric operation $\{\cdot,\cdot\}\colon\mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ satisfying Leibniz's rule in each argument and Jacobi's identity.

There is an alternative and useful characterization of Poisson brackets. Given any bracket $\{\cdot,\cdot\}\colon\mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ satisfying Leibniz's rule and linearity in each variable, we can recover its action on any functions $f,g\in\mathcal{C}^\infty(M)$ as the contraction of a two-tensor field $\Pi\in\mathcal{T}^2M$ with the differentials $\mathrm{d}f$, $\mathrm{d}g$. Moreover, because the Poisson bracket is skew-symmetric, there exists a unique bivector field $\Pi\in\mathfrak{X}^2(M)$ representing the bracket $\{\cdot,\cdot\}$ in the sense that

$$\{f,g\}=\langle\mathrm{d}f\wedge\mathrm{d}g,\Pi\rangle$$

for any smooth functions $f,g\in\mathcal{C}^\infty(M)$. Jacobi's identity, however, does not hold for general bivector fields. It turns out to be equivalent the integrability condition $\mathcal{J}=[\Pi,\Pi]=0$. The trivector field $\mathcal{J}$, up to a factor, is appropriately called the *Jacobiator*, and the bracket $[\cdot,\cdot]$ is an extension of the Lie bracket of vector fields to the space of all multivector fields called *Schouten–Nijenhuis bracket*.

Given the great generality of these structures, there is no local normal form for Poisson structures similar to Darboux's Theorem 2.2. The closest analogue is the following result due to Weinstein.

Theorem 2.4 (Weinstein [15])**.** *Let (M,Π) be a Poisson manifold. For every point $p\in M$ there exists a chart $\varphi\colon U\subseteq M\to\mathbb{R}^n$ with coordinates q_i, p_j, r_k such that*

$$\varphi_*\Pi=\sum_{i=1}^{k}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}+\sum_{i,j=1}^{n-2k}f_{ij}(r_l)\frac{\partial}{\partial r_i}\wedge\frac{\partial}{\partial r_j}.$$

Moreover, the functions f_{ij} are skew-symmetric and vanish at 0.

This local structure theorem is commonly called the *splitting theorem* because it states that, locally, every Poisson manifold splits as the direct product of a symplectic manifold and a Poisson manifold with vanishing Poisson structure at the origin. Observe this transverse Poisson structure measures the difference of a Poisson manifold from being symplectic.

We would like to highlight two immediate consequences from Weinstein's theorem. Firstly, the splitting shows that any Poisson manifold admits a foliation by symplectic leaves, called the *symplectic foliation* of the manifold. This shows that part of the Poisson structure can be encoded in the symplectic structures of the leaves. Secondly, there is a well-defined notion of *transverse* Poisson structure. In contrast with the symplectic realm, Poisson manifolds do have local invariants. As such, their study is much more complicated.

3. Singular symplectic geometry

We have seen that the class of symplectic manifolds fits very naturally within the class of Poisson manifolds. The class of Poisson manifolds is, however, much bigger and wilder than that of symplectic manifolds. As such, there are some interesting and nice results in the symplectic category that do not hold in Poisson geometry. One instance of this phenomenon is hinted in the difference between Darboux's Theorem 2.2 and Weinstein's Theorem 2.4.

There are many specific examples of Poisson manifolds which, although not being symplectic, can be understood in a symplectic flavour if we are willing to work with singularities. Take as an example the simplest degenerate Poisson structure with its dual form,

$$\Pi = z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i},$$

$$\omega = \frac{\mathrm{d}z}{z} \wedge \mathrm{d}t + \sum_{i=2}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

The form ω is clearly not a symplectic form, because it is not even well-defined as a smooth differential form. It becomes a symplectic form, in some sense, if we restrict its domain to the space of vector fields tangent to the hypersurface $\{z = 0\}$.

We can informally call ω a singular symplectic form. The objective of this section is to elevate this idea to a rigorous statement. We begin by defining the main objects of the discussion.

Definition 3.1. A *Lie algebroid* is a vector bundle $\pi\colon \mathcal{A} \to M$ together with a vector bundle map $\rho\colon \mathcal{A} \to \mathrm{T}M$ covering the identity and equipped with a Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ on the space of sections $\Gamma\mathcal{A}$. Moreover, the bracket satisfies, for any $X, Y \in \Gamma A$ and $f \in \mathcal{C}^\infty(M)$, the following compatibility conditions:

$$[X, fY]_{\mathcal{A}} = f[X, Y]_{\mathcal{A}} + \mathcal{L}_{\rho(X)} f \cdot Y, \tag{5a}$$

$$\rho([X, Y]_{\mathcal{A}}) = [\rho(X), \rho(Y)]. \tag{5b}$$

In equation (5a), the operator $\mathcal{L}$ denotes the Lie derivative of a function along a vector field.

Equation (5a) is a generalized Leibniz's identity for the bracket. Equation (5b) turns out to be redundant, as it can be deduced from (5a). We have chosen to explicitly state it because it will be relevant for upcoming discussions.

Let us take a brief detour and precisely describe how these objects arise in the description of systems with singularities. We will present examples arising from physics where all the following assumptions are satisfied. Consider that the equations of motion of our system can be described in terms of a $\mathcal{C}^\infty$ subsheaf of vector fields $\mathcal{F} \subseteq \mathfrak{X}$. Furthermore, assume the sheaf is locally finitely generated, that is, for any point $p \in M$ there is an open set U containing p and sections $X_1, \dots, X_m \in \mathcal{F}_U$ such that their restriction to any open set $V \subseteq U$ generates $\mathcal{F}_V$. We can make two additional assumptions, each of which gives rise to well-known objects in differential geometry.

- If we additionally assume the integrability condition $[\mathcal{F}, \mathcal{F}] = \mathcal{F}$, the sheaf $\mathcal{F}$ defines a *singular foliation* in the sense of Androulidakis and Skandalis. These objects can be integrated to give standard singular foliations, or foliations in the sense of Stefan and Sussman.

- If the sheaf is not only locally finitely generated but also locally free, it is a theorem of Serre [11] in the algebraic setting and Swan [13] in the continuous case shows the sheaf $\mathcal{F}$ can be recovered as the sheaf of sections of a vector bundle E.

If both assumptions are simultaneously made, we get *projective foliations* or *Debord foliations*. If E is a representing vector bundle for $\mathcal{F}$ in the sense that $\mathcal{F} = \Gamma E$, we get a natural map of vector bundles $\rho\colon E \to \mathrm{T}M$ given by the evaluation of a section at a point. This map is called the anchor, and is injective in an open and dense subset $U \subseteq M$, i.e., generically injective. The integrability condition $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$ lifts to a bracket in the space of sections ΓE. One can easily check that the compatibility conditions (5a) and (5b) are satisfied and, thus, any such object is an instance of a Lie algebroid.

Not all Lie algebroids arise this way as, in general, the anchor map $\rho\colon \mathcal{A} \to \mathrm{T}M$ is not generically injective. The class of algebroids previously presented will be relevant in upcoming sections, so we will give them a proper name.

Definition 3.2. Let M be a smooth manifold. An *E-structure* is the choice of a Debord foliation $\mathcal{F} \subseteq \mathfrak{X}$ or, equivalently, a vector bundle $\pi\colon E \to M$ with a generically injective map $\rho\colon E \to \mathrm{T}M$. We call the pair (E, M) an *E-manifold*.

This construction shows that we can consider Lie algebroids, at least psychologically, as a replacement of the standard tangent bundle $\mathrm{T}M$. As such, we can consider the dual bundle $\mathcal{A}^*$ and its exterior powers $\bigwedge^k \mathcal{A}^*$. Sections of these bundles are called, by analogy with the standard setting, *k-differential $\mathcal{A}$-forms*. The space of all sections is written $\Omega^k_{\mathcal{A}}(M)$. The Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ induces an exterior differential in the spaces $\Omega^k_{\mathcal{A}}(M)$ following the standard Koszul formula,

$$\begin{aligned}\mathrm{d}_{\mathcal{A}}\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k}(-1)^i \mathcal{L}_{\rho(X_i)}\omega(X_0, \dots, \hat{X}_i, \dots, X_k) \\ &+ \sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k).\end{aligned}$$

In the previous formula, an argument with a hat implies it has been suppressed from the collection of inputs. A routine verification shows $\mathrm{d}^2_{\mathcal{A}} = 0$. The cohomology spaces of the complex of $\mathcal{A}$-forms are called *Lie algebroid cohomology groups*.

With the notion of differential forms and exterior calculus for systems with constraints, we can define a symplectic form mimicking the standard definition in classical differential geometry.

Definition 3.3. Let $\pi\colon \mathcal{A} \to M$ be a Lie algebroid. A *symplectic form* on $\mathcal{A}$ is a two form $\omega \in \Omega^2_{\mathcal{A}}(M)$ which is closed and non-degenerate. We call the pair $(\mathcal{A}, \omega)$ a *symplectic Lie algebroid*. Similarly, if $\pi\colon E \to M$ is an E-manifold, we call the pair (E, ω) an *E-symplectic manifold*.

In this setting, the non-degeneracy condition amounts to requiring the vector bundle morphism

$$\begin{aligned}\omega^\flat\colon \mathcal{A} &\longrightarrow \mathcal{A}^* \\ X &\longmapsto \iota_X\omega\end{aligned}$$

to be an isomorphism. Its inverse map is written $\omega^\sharp\colon \mathcal{A}^* \to \mathcal{A}$. As a consequence, we can define the Hamiltonian vector field associated to a function H as the unique solution to the equation

$$\iota_{X_H}\omega = -\rho^\vee \mathrm{d}H.$$

In the previous equation, the map $\rho^\vee\colon \mathrm{T}^*M \to \mathcal{A}^*$ denotes the adjoint of the anchor map $\rho\colon \mathcal{A} \to \mathrm{T}M$.

We motivated the construction of Lie algebroids and symplectic forms on them to study Poisson structures with certain types of singularities. Any symplectic Lie algebroid indeed defines a Poisson bracket as

$$\{f, g\} = \omega(X_f, X_g).$$

This mapping is clearly bilinear, skew-symmetric, and satisfies Leibniz's identity. Jacobi's identity is not as evident but it is a consequence of the closedness of ω as a singular symplectic form.

Before concluding this section, let us briefly discuss two different examples of E-manifolds which have found success in Poisson geometry.

We begin by describing *b-symplectic manifolds*. In this case, the sheaf of vector fields considered is taken to be the sheaf of tangent vectors to an embedded hypersurface $Z \subseteq M$. The Lie algebroid obtained is called the *b-tangent bundle*. It was originally considered by Melrose [5] in order to generalize the index theorem to manifolds with boundary. The symplectic geometry of b-manifolds has been extensively studied and described by Guillemin, Miranda, and Pires [2]. The school of Miranda has done remarkable work in studying the interplay of b-symplectic geometry with integrable systems, geometric quantization, KAM theory, and many more.

These objects are generalized by b^m-symplectic manifolds. The sheaf of vector fields into consideration is once again the sheaf of all fields tangent to a fixed hypersurface Z, but now we fix with degree of tangency to be at least m. The definition of these structures is due to Scott [10], where some technical details concerning additional data are discussed. This singular symplectic models have found applications in studying the topology of escape orbits in the planar, restricted, circular three body problem [7].

4. Reduction by symmetries and minimal coupling

One of the central ideas in the study of physical systems is that of symmetries. In the presence of a group of transformations that leaves the motion of the system unchanged, one can reduce the number of parameters by an appropriate choice of coordinates (or frames of reference). A remarkable instance of this phenomenon is Euler's solution to the two-body problem. The invariance by linear translations allows the origin of the frame of reference to be taken in the center of mass, while the invariance by rotations implies the confinement of both bodies to a plane and one additional constraint.

These invariance by transformation groups of the system can be dually read as conservation laws. The invariance by linear transformations is equivalent to the conservation of linear momentum, while the invariance by rotations is equivalent to the conservation of angular momentum. The observation that this phenomenon is a general feature is due to Emmy Noether and, as such, the conserved quantities obtained from a symmetry are called *Noether charges*.

This correspondence is transparent in the symplectic formulation of classical mechanics. To describe it, we will need to define what does it mean for an action of a Lie group G on (M, ω) to be Hamiltonian. Intuitively, we would like the fundamental vector fields of the action to be Hamiltonian: in other words, we are asking for a lift $\mu^\bullet$ of the fundamental vector field map $\bullet^\#$ in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{C}^\infty(M) & \xrightarrow{X_\bullet} & \mathfrak{X}_{\mathrm{Ham}}(M) & \longrightarrow & 0 \\ & & & & & \nwarrow^{\mu^\bullet} & \uparrow{\scriptstyle \bullet^\#} & & \\ & & & & & & \mathfrak{g} & & \end{array} \tag{6}$$

Here, we have tacitly assumed M is connected so that $\mathrm{H}^0(M) = \mathbb{R}$. There always exists such a lift at the level of vector spaces: as $X_\bullet \colon \mathcal{C}^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M)$ is surjective, we can choose the preimage of a set of generators and extend by linearity.

There are obstructions for the map $\mu^\bullet \colon \mathfrak{g} \to \mathcal{C}^\infty(M)$ to be a morphism of Lie algebras, where we endow $\mathcal{C}^\infty(M)$ with the Poisson bracket as Lie bracket. The failure to have a Lie algebra morphism is measured by the map

$$c(X, Y) = \{\mu^X, \mu^Y\} - \mu^{[X,Y]}.$$

As the projection of this element to $\mathfrak{X}_{\mathrm{Ham}}(M)$ vanishes by the commutativity of the diagram, we can identify the image $c(X, Y)$ with an element in the kernel $\ker X_\bullet \simeq \mathbb{R}$. This map thus determines an element in the Chevalley–Eilenberg complex, $c \in \mathrm{C}(\mathfrak{g}; \mathbb{R})$. The map c is closed, and hence determines a class in the cohomology group $\mathrm{H}^2(\mathfrak{g}; \mathbb{R})$. The lift $\mu^\bullet$ can be chosen to be a Lie algebra morphism if and only if $[c] = 0$. Moreover, all possible such choices are parametrized by elements of the group $\mathrm{H}^1(\mathfrak{g}; \mathbb{R})$.

Assuming some conditions on the Lie group G,[2] there is a uniquely determined lift which we call the *comoment map*. By construction, it intertwines the adjoint action in $\mathfrak{g}$ with the induced pullback action in $\mathcal{C}^\infty(M)$. As the name hints, however, it is better to think of the comoment map in terms of a dual object called the *moment map*.

Definition 4.1. Let (M, ω) be a symplectic manifold and let G be a Lie group acting on M. We say the action is *Hamiltonian* if there exists a map $\mu \colon M \to \mathfrak{g}^*$, called the *moment map*, satisfying the following conditions:

$$\iota_{X^\#}\omega_p = -\mathrm{d}\langle \mu(p), X\rangle \quad \text{for all } X \in \mathfrak{g},\ p \in M, \tag{7a}$$

$$\mu \circ \rho_g = \mathrm{Ad}^*_g \circ \mu \qquad \text{for all } g \in G. \tag{7b}$$

We call any such triple (M, ω, μ) a *Hamiltonian G-space*.

Equation (7a) is reminiscent of the equivariance of the comoment map. Equation (7b) is a direct consequence of the fact that the fundamental vector fields of $\mathfrak{g}$ act in a Hamiltonian fashion: indeed, all we are saying is that $\langle \mu, X\rangle$ is the Hamiltonian function of $X^\#$ or, following (6), the comoment map μ^X.

Let us assume now we are given a Hamiltonian G-space (M, ω, μ) with a G-invariant Hamiltonian H and equations of motion determined by X_H. Assume H is an invariant function under the action of G. We will also assume the group G is connected. This technical condition ensures that any Hamiltonian action is also a symplectic action, that is, preserves the form ω. Two consequences arise from these facts.

Firstly, observe X_H is G-invariant. We recall the symplectic form ω is G-invariant because we assume G is connected. Because the Hamiltonian H is G-invariant, we have

$$\iota_{\rho_g \cdot X_H}\omega = \rho^*_{g^{-1}}\iota_{X_H}\rho^*_g\omega = \rho^*_{g^{-1}}\mathrm{d}H = \mathrm{d}H = \iota_{X_H}\omega.$$

As a consequence, we deduce $X_H = \rho_g \cdot X_H$.

Secondly, assume we are given a regular value $\alpha \in \mathfrak{g}^*$ of μ. By equivariance, every other value $\beta \in \mathcal{O}_\alpha$ is regular, and thus the preimage $M_\alpha = \mu^{-1}(\mathcal{O}_\alpha)$ is a submanifold. By invariance of H we have $\mathcal{L}_{X^\#}H = 0$. The definition of moment map implies now

$$0 = \mathcal{L}_{X_H}\mu^X = \langle \mathrm{d}\mu^X, X_H\rangle = \langle \iota_{X_H}\mu, X\rangle.$$

[2] For example, if G is semisimple, we know by Whitehead's lemmas that $\mathrm{H}^1(\mathfrak{g}; \mathbb{R}) = \mathrm{H}^2(\mathfrak{g}; \mathbb{R}) = 0$. The semisimplicity assumption is automatically satisfied if G is a compact group.

As this equality holds for all $X \in \mathfrak{g}$, we have $\langle \mathrm{d}\mu, X_H \rangle = 0$. This result implies X_H is tangent to the level sets of μ and, by G-invariance, to the preimages of the coadjoint orbits.

Thus, the dynamics defined by X_H can be restricted to the submanifold $\mu^{-1}(\mathcal{O}_\alpha)$ and can be further projected to the quotient manifold $\mu^{-1}(\mathcal{O}_\alpha)/G_\alpha$, assuming technical conditions on the action of G.[3] Marsden and Weinstein observed that this reduced space is once again a symplectic manifold, and hence one can consider Hamiltonian dynamics with respect to the reduced symplectic structure.

Theorem 4.2 (Marsden–Weinstein [4]). *Let (M, ω) be a symplectic manifold and assume G is a compact Lie group acting on M with moment map $\mu\colon M \to \mathfrak{g}^*$. If $\alpha \in \mathfrak{g}^*$ is a regular value of μ, then the space $\mu^{-1}(\mathcal{O}_\alpha)/G_\alpha$ is a symplectic manifold with symplectic form ω_{red}. Moreover, it is uniquely determined by*

$$i_\alpha^* \omega = \pi^* \omega_{\mathrm{red}}.$$

4.1 The minimal coupling procedure

We will describe a procedure to construct the phase space of a charged particle interacting with a Yang–Mills field. The presentation we take is essentially due to Sternberg [12]. Assume we are given a principal G-bundle $\pi\colon P \to X$ over a symplectic manifold (X, ω) and a Hamiltonian G-space (Q, Ω) with moment map μ. We can construct a symplectic structure in the adjoint bundle $P \times_G Q$ by choosing a principal connection $\eta \in \Omega^1(X; \mathrm{ad}\, P)$ in the following way. Sternberg shows the two-form $\mathrm{d}\langle \mu, \eta \rangle + \Omega$ in $P \times Q$ descends to a well-defined and closed two-form $\Omega_\eta \in \Omega^2(P \times_G Q)$. Under a non-degeneracy assumption, the manifold $(P \times_G Q, \omega + \Omega_\eta)$ is symplectic and the previous construction is called the *minimal coupling procedure*. The additional term Ω_η is known in the literature as the *magnetic term*. If we take (Q, Ω) to be a coadjoint orbit of an irreducible representation of a Lie group G, we obtain the classical phase spaces of charged particles [12].

Sternberg mentions that, in the case where $X = \mathrm{T}M$ with its canonical symplectic form, the previous non-degeneracy assumption is always satisfied. Weinstein went beyond this observation and proved Sternberg's phase space can be obtained as the symplectic reduction of a universal phase space. The role of the connection is made explicit in terms of an isomorphism between his construction and Sternberg's. More concretely, we can summarize Weinstein's results in the following theorem.

Theorem 4.3 (Weinstein [14]). *Let $\pi\colon P \to M$ be a principal G-bundle and let (Q, Ω) be a Hamiltonian G-space with moment map μ_Q. Let $P^\#$ be the pullback bundle of π by the submersion $\mathrm{T}^*M \to M$.*

Then, the space $\mathrm{T}^*P \times Q$ is a G-Hamiltonian space for the diagonal G-action with moment map[4] $\mu = \mu_P + \mu_Q$. Any choice of connection in P induces a diffeomorphism $\mu^{-1}(0) \simeq P^\# \times Q$ which, furthermore, induces a diffeomorphism of the symplectic spaces $\mu^{-1}(0)/G$ and $P^\# \times_G Q$.

The symplectomorphism induced by this choice of connection is called the *minimal coupling* of the system. Given a Hamiltonian in the base space, $H \in \mathcal{C}^\infty(\mathrm{T}^*M)$, we can consider its pullback to either space and get equivalent dynamics. The induced equations of motion are called *Wong's equations* [8].

[3] Namely, freeness and properness. The latter is satisfied if G is a compact group, which we have enforced since the construction of the moment map.

[4] In this formula, the moment map $\mu_P\colon \mathrm{T}^*P \to \mathfrak{g}^*$ is the natural moment map obtained for the cotangent lift of any G-action to the cotangent bundle with the canonical symplectic form.

There is an interesting interpretation of this construction by Montgomery [9]. The choice of a connection yields the commutative diagram

$$\begin{array}{ccc} P^{\#} \times_G Q & \xrightarrow{\Phi} & \mu^{-1}(0)/G \\ & \searrow^{\pi} & \downarrow{\scriptstyle h^{\vee}} \\ & & \mathrm{T}^*M \end{array}$$

Here, the projection π is completely natural and is induced from the projection of the pullback bundle $\pi\colon P^{\#} \to \mathrm{T}^*M$. The map $h^{\vee}$ is the dual of the horizontal lift $h\colon \mathrm{T}M \to \mathrm{T}P$, completely determined and equivalent to the choice of connection. Therefore, we have two different ways to understand Wong's equations of motion. In the space $P^{\#} \times_G Q$, the Hamiltonian function does not get modified but the symplectic structure absorbs the additional factor Ω_η. In the universal model $\mu^{-1}(0)/G$, the symplectic form is canonical but the Hamiltonian function gets twisted by the pullback under $h^{\vee}$.

4.2 The singular minimal coupling

The minimal coupling enables the study of classical particles interacting with Yang–Mills field in the symplectic formulation. The extension of this construction to include systems modeled with E-manifolds was proposed by Mir, Miranda, and Nicolás [6]. More concretely, one of the results proved is the following analogous statement to Theorem 4.3.

Theorem 4.4 (Mir–Miranda–Nicolás [6])**.** *Let $\pi\colon P \to M$ be a principal G-bundle over an E-manifold $E \to M$ and let (Q, Ω) be a Hamiltonian G-space with moment map μ_Q. Let $P^{\#}$ be the pullback bundle of π by the submersion $E_M^* \to M$.*

Then, the space $E_P^* \times Q$ is a G-Hamiltonian space for the diagonal G-action with moment map $\mu = \mu_P + \mu_Q$. Any choice of connection in P induces a diffeomorphism $\mu^{-1}(0) \simeq P^{\#} \times Q$ which, furthermore, induces a diffeomorphism of the symplectic spaces $\mu^{-1}(0)/G$ and $P^{\#} \times_G Q$.

Throughout the rest of the section we will fix an E-manifold $E_M \to M$ and a principal G-bundle $P \to M$. The proof of Theorem 4.4 follows essentially the same argument as Weinstein. The complication lies in developing the machinery necessary to state and follow the original proof. Since we would like to extend the singularities of our configuration space M to the bundles E_M and P, we need a procedure to do so. The fundamental notion is that of *prolongation*, which dates back at least to the works of de León, Marrero, and Martínez [1].

Definition 4.5. Assume $f\colon N \to M$ is a surjective submersion over a Lie algebroid $\mathcal{A} \to M$. The *prolongation* of A along f, written $\mathcal{L}^f\mathcal{A}$, is the pullback bundle of the morphisms $\mathrm{d}f\colon \mathrm{T}N \to \mathrm{T}M$ and $\rho\colon \mathcal{A} \to M$. As a set, it can be identified with

$$\mathcal{L}^f\mathcal{A} = \{(X, Y) \in \mathcal{A} \times \mathrm{T}N \mid \rho(X) = \mathrm{d}f(Y)\}.$$

Throughout the rest of the section, we fix an E-manifold $E \to M$. The prolongation of the dual bundle $E_M^* \to M$, which can be thought as the singular tangent bundle of the cotangent bundle, carries a natural Liouville form whose differential is symplectic [1]. Thus, the prolongation of E_M^* is a symplectic E-manifold, in strong resemblance to the cotangent bundle of a smooth manifold.

Similarly, we consider the prolongation of the principal G-bundle $P \to M$. Because $E_P \to P$ has a natural action of the Lie group G and the anchor map is injective on an open and dense subset, the action on P lifts to an action on E_P which factors through the standard tangent map. By duality, the action lifts to the dual bundle E_P^* and, moreover, it becomes Hamiltonian with respect to the canonical symplectic structure. The fact that the action of the Lie group G automatically lifts to E_P is only valid for E-manifolds. If we want to establish similar results for symplectic Lie algebroids, stronger compatibility assumptions are needed to define Hamiltonian group actions (see [3]).

The last ingredient in the proof of Theorem 4.4 is the notion of symplectic reduction in the singular setting. The authors rely on a version of the reduction theorem developed by Marrero, Padrón, and Rodríguez-Olmos for symplectic Lie algebroids (Theorem 3.11 in [3]).

In [6] the authors consider some standard configuration spaces with singularities, such as the compactification of a stationary black hole or a general b^m-manifold, motivated by previous contributions in the literature of celestial mechanics. Moreover, they explicitly compute Wong's equations describing the motion of a charged particle interacting with a Yang–Mills field.

5. Conclusions

Symplectic manifolds are fundamental objects in the geometric formulation of Hamiltonian dynamics. These give rise to Poisson brackets, which measure the evolution of observables along the trajectories of the system but are vastly more general. E-Symplectic manifolds lie between both worlds: even though they define Poisson structures, their behavior is closer to symplectic forms. Moreover, they naturally encode certain physical systems with constrained dynamics.

Theorem 4.4 extends the classical minimal coupling procedure to E-symplectic manifolds. In more physical terms, it provides a Hamiltonian formulation of the equations of motion of particles under the interaction with a Yang–Mills field for constrained systems. This result could open the door to study the dynamics of such physical systems using geometric techniques. Indeed, in [7] the authors obtain a b^3-symplectic structure in the planar, restricted, circular three-body problem and, using a contact analogue of the theory described here, discuss the existence of periodic orbits at infinity. No analogue result has been established for charged particles.

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On the behaviour of Hodge spectral exponents of plane branches

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Resum *(CAT)*
Els exponents espectrals de Hodge són un conjunt discret d'invariants d'una singularitat aïllada d'una hipersuperfície. En aquest article estudiem la seva distribució pel cas de branques planes, en termes d'invariants numèrics de la branca. En primer lloc calculem la distribució límit per a diferents maneres de fer el límit. En segon lloc donem una fórmula tancada per a la diferència acumulada entre la distribució d'exponents espectrals de Hodge i una distribució contínua, la qual és el límit més comú. Utilitzem aquesta expressió per a obtenir intervals de valors dominants.

Abstract *(ENG)*
The Hodge spectral exponents are a discrete set of invariants of an isolated hypersurface singularity. We study their distribution for the case of plane branches, in terms of numerical invariants of the branch. First, we calculate the limit distribution for different ways of taking the limit. Secondly, we provide a closed formula for the cumulative difference of the distribution of Hodge spectral exponents with a continuous distribution, which is the most common limit. We use this expression to obtain intervals of dominating values.

Keywords: *Hodge spectral exponents, plane curve singularities, Puiseux pairs, limit distribution, characteristic function, jumping numbers.*
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1. Introduction

Let $f\colon (\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of a holomorphic function (or equivalently a convergent power series $f\in\mathbb{C}\{x_0,\dots,x_n\}$) with an isolated singularity at the origin. Using the canonical mixed Hodge structure of the cohomology groups of the Milnor fiber of f, Steenbrink [7] defined the *Hodge spectrum* of f as the generating function

$$\mathrm{Sp}_f(T)=\sum_{i=1}^{\mu}T^{\alpha_i},$$

where $\mu=\dim_{\mathbb{C}}\frac{\mathbb{C}\{x_0,\dots,x_n\}}{\left(\frac{df}{dx_0},\dots,\frac{df}{dx_n}\right)}$ is the *Milnor number* and the positive rational numbers

$$0<\alpha_1\leqslant\cdots\leqslant\alpha_\mu<n+1$$

form a discrete set of invariants of the singularity f called *Hodge spectral exponents* (or *spectral numbers*). They are symmetric with respect to $(n+1)/2$, i.e., for every $j=1,\dots,\mu$, we have $\alpha_{\mu+1-j}=(n+1)-\alpha_j$ and thus it is enough to study them in the interval $(0,(n+1)/2]$.

Another interesting feature proved by Varchenko [10] is that the Hodge spectral exponents of f are stable under deformations with constant Milnor number μ. A deformation of a hypersurface $f(x_0,\dots,x_n)\in\mathbb{C}\{x_0,\dots,x_n\}$ is a family of hypersurfaces $f_{t_1,\dots,t_k}(x_0,\dots,x_n)$ for some set of parameters $(t_1,\dots,t_k)\in S\subseteq\mathbb{C}^k$, satisfying $f(x_0,\dots,x_n)=f_{0,\dots,0}(x_0,\dots,x_n)$. Then, in Varchenko's result we are asking that the Milnor number of $f_{t_1,\dots,t_k}(x_0,\dots,x_n)$ is the same for all $(t_1,\dots,t_k)\in S$.

K. Saito [4] considered the normalized spectrum which he denoted as the *characteristic function*

$$\chi_f(T)=\frac{1}{\mu}\sum_{i=1}^{\mu}T^{\alpha_i}.$$

We may also display the Hodge spectral exponents as a discrete (probability) distribution on $\mathbb{R}$. Namely, the *distribution of the Hodge spectral exponents* is

$$D_f(s)=\frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i),$$

where $\delta(s)$ is the Dirac's delta distribution. Indeed, considering either $D_f(s)$ or $\chi_f(T)$ is equivalent because the characteristic function is the Fourier transform of the distribution of Hodge spectral exponents, i.e.,

$$\chi_f(T)=\mathcal{F}\{D_f(s)\}(\tau).$$

Considering the change of variables $T=e^{2\pi\mathrm{i}\tau}$ we treat the dependence of χ_f on T and on τ interchangeably throughout this paper.

Remark 1.1. Because of the symmetry of the Hodge spectrum, we are interested in the truncations

$$\chi_f^{<1}(T)=\frac{1}{\mu}\sum_{\alpha_i<1}T^{\alpha_i},\quad D_f^{<1}(s)=\frac{1}{\mu}\sum_{\alpha_i<1}\delta(s-\alpha_i).$$

Definition 1.2. The *continuous distribution* function is $N_{n+1}\colon \mathbb{R} \to \mathbb{R}$ defined as:

$$N_{n+1}(s) = \int_{x_0+\cdots+x_n=s} \mathbb{1}_{[0,1)}(x_0)\cdots\mathbb{1}_{[0,1)}(x_n)\,\mathrm{d}x_0\ldots\mathrm{d}x_n = \big(\mathbb{1}_{[0,1)} * \overset{n+1}{\cdots} * \mathbb{1}_{[0,1)}\big)(s),$$

where $\mathbb{1}_{[0,1)}(s)$ is the indicator function and $*$ denotes the convolution product. One may check that the Fourier transform of $N_{n+1}(s)$ is

$$\mathcal{F}\{N_{n+1}(s)\}(\tau) = \left(\frac{T-1}{\log T}\right)^{n+1}.$$

Definition 1.3. We define $\phi_f\colon \left[0, \frac{n+1}{2}\right) \to \mathbb{R}$ as the *cumulative difference function* between $N_{n+1}(s)$ and $D_f(s)$, that is,

$$\phi_f(r) = \int_0^r N_{n+1}(s) - D_f(s)\,\mathrm{d}s = \int_0^r N_{n+1}(s) - \frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)\,\mathrm{d}s = \int_0^r N_{n+1}(s)\,\mathrm{d}s - \frac{1}{\mu}\#\{\alpha_i \leqslant r\}.$$

Definition 1.4. We say that $r \in \left[0, \frac{n+1}{2}\right)$ is a *dominating value* if $\phi_f(r) > 0$, or equivalently if

$$\frac{1}{\mu}\#\{\alpha_i \leqslant r\} < \int_0^r N_{n+1}(s)\,\mathrm{d}s.$$

K. Saito [4] introduced these notions of cumulative difference function and dominating values. Moreover he formulated the following questions:

Question 1. *For which limits of sequences of hypersurfaces $(f^{(i)})_{i\geqslant 0}$ does the distribution of Hodge spectral exponents $D_{f^{(i)}}(s)$ converge to $N_{n+1}(s)$? Equivalently, for which limits of $(f^{(i)})_{i\geqslant 0}$ does the characteristic function $\chi_{f^{(i)}}(T)$ converge to $\mathcal{F}\{N_{n+1}\}(\tau) = \left(\frac{T-1}{\log T}\right)^{n+1}$?*

Question 2. *Given f, what is the set of all dominating values?*

The limit of $(f^{(i)})_{i\geqslant 0}$ in Question 1 has to be specified, since it is not clear a priori which kind of limit one should consider. The few results we may find in the literature all consider different types of limits. K. Saito already calculated the following two limits, both of which converge to $N_{n+1}(s)$:

Proposition 1.5 ([4, (3.7)])**.** *Let $f \in \mathbb{C}[x_0,\ldots,x_n]$ be a quasi-homogeneous polynomial of degree 1 with respect to the weights $r_0,\ldots,r_n$, i.e., satisfying $f(\lambda^{r_0}x_0,\ldots,\lambda^{r_n}x_n) = \lambda f(x_0,\ldots,x_n)$. Then, taking a sequence of such functions with the limit $r_i \to 0$ for all $i = 0,\ldots,n$, one has*

$$\lim_{r_0,\ldots,r_n\to 0}\chi_f(T) = \left(\frac{T-1}{\log T}\right)^{n+1} = \mathcal{F}\{N_{n+1}(s)\}(\tau).$$

Proposition 1.6 ([4, (3.9)])**.** *Let $f\in\mathbb{C}\{x,y\}$ be an irreducible plane curve with Puiseux pairs $(n_1,l_1),\ldots,(n_g,l_g)$. Then, taking a sequence of such functions with the limit $n_g \to +\infty$ (keeping all other n_j and l_j fixed), one has*

$$\lim_{n_g\to+\infty}\chi_f(T) = \left(\frac{T-1}{\log T}\right)^{2} = \mathcal{F}\{N_2(s)\}(\tau).$$

The Puiseux pairs are defined in Section 2.

More recently, Almirón and Schulze gave another example for which the distribution of Hodge spectral exponents also converges to the continuous distribution $N_{n+1}(s)$:

Proposition 1.7 ([1]). *Consider a fixed Newton diagram Γ. Let $f_\omega \in \mathbb{C}\{x_0, \dots, x_n\}$ be a Newton non-degenerate function with Newton diagram $\omega\Gamma$ (the rescaling of Γ by a factor $\omega \in \mathbb{Q}_{>0}$). Then, taking a sequence of such functions with limit $\omega \to +\infty$, one has*

$$\lim_{\omega\to+\infty} \chi_{f_\omega}(T) = \left(\frac{T-1}{\log T}\right)^{n+1} = \mathcal{F}\{N_{n+1}(s)\}(\tau).$$

Regarding Question 2 on the set of dominating values, Tomari proved the following result which, in terms of dominating values, states the following:

Theorem 1.8 ([9]). *Let $f \in \mathbb{C}\{x, y\}$ be a plane curve. Then $\frac{1}{2}$ is a dominating value, i.e.,*

$$\#\left\{\alpha_i \leqslant \frac{1}{2}\right\} < \frac{\mu}{8}.$$

K. Saito asked whether $\frac{1}{2}$ is a dominating value for any $f \in \mathbb{C}\{x_0, \dots, x_n\}$, that is, whether

$$\#\left\{\alpha_i \leqslant \frac{1}{2}\right\} < \frac{\mu}{(n+1)!\,2^{n+1}}.$$

A conjecture posed by Durfee states:

Conjecture 1.9 ([3]). *Let $f \in \mathbb{C}\{x, y, z\}$ be a surface with a singularity at the origin. Then*

$$p_g < \frac{\mu}{6}.$$

Here, p_g denotes the *geometric genus* of f defined as

$$p_g = \dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_X)_0 \text{ for } n \geqslant 2 \quad (p_g = \dim_{\mathbb{C}}(\pi_*\mathcal{O}_X/\mathcal{O}_{\mathbb{C}^2})_0 \text{ for } n = 1),$$

with $\pi\colon X \to \mathbb{C}^{n+1}$ being a resolution of the singularity. M. Saito [5] proved a relation between this invariant and the Hodge spectral exponents, namely $p_g = \#\{i \mid \alpha_i \leqslant 1\}$, and thus Durfee's conjecture predicts that 1 is a dominating value for $n = 2$. K. Saito asked whether one can generalize this statement:

Question 3. *Is 1 a dominating value for all $n \geqslant 2$? That is, is it true that*

$$p_g = \#\{\alpha_i \leqslant 1\} < \frac{\mu}{(n+1)!}$$

for any $f \in \mathbb{C}\{x_0, \dots, x_n\}$?

The aim of this work is to study Questions 1 and 2 for the case of plane branches. Regarding Question 1 we calculate the limit distribution for the limits $n_k \to +\infty$ and $l_k \to +\infty$. Regarding Question 2, we give a closed formula for $\#\{\alpha_i \leqslant r\}$ in Theorem 4.1 and $\phi_f(r)$ in Theorem 4.2 in terms of numerical invariants of the plane branch. Thereby, we provide in Theorem 5.1 intervals of dominating values.

2. Plane branch singularities

In this section we briefly present the necessary background on irreducible plane curves that we use in this paper and we refer to Casas-Alvero's book [2] for unexplained terminology.

Let $f\colon (\mathbb{C}^2,0)\to(\mathbb{C},0)$ be a germ of a holomorphic function, or equivalently a convergent power series $f\in\mathbb{C}\{x,y\}$. The equation $f=0$ defines locally a (complex) plane curve around the origin. We only consider irreducible plane curves f (also called plane branches), i.e., irreducible elements of the unique factorization domain $\mathbb{C}\{x,y\}$.

Theorem 2.1 (Puiseux). *Let $f\in\mathbb{C}\{x,y\}$ define an irreducible plane curve that is not tangent to the y-axis (i.e., $\frac{\partial f}{\partial x}\neq 0$). Then there is a Puiseux series $s(x)=\sum_{i\geqslant 0}a_i x^{i/m}$ such that $f(x,s(x))=0$. Moreover, all such series are conjugates $\sigma_\varepsilon(s)=\sum_{i\geqslant 0}a_i\varepsilon^i x^{i/m}$ with $\varepsilon^m=1$. The curve can be parameterized by $t\mapsto\left(t^m,\sum_{i\geqslant 0}a_i t^i\right)$.*

A Puiseux series of f has the form

$$s(x)=\sum_{\substack{j\in(e_0)\\0\leqslant j<\beta_1}}a_j x^{j/m}+\sum_{\substack{j\in(e_1)\\\beta_1\leqslant j<\beta_2}}a_j x^{j/m}+\cdots+\sum_{\substack{j\in(e_{g-1})\\\beta_{g-1}\leqslant j<\beta_g}}a_j x^{j/m}+\sum_{\substack{j\in(e_g)\\\beta_g\leqslant j}}a_j x^{j/m}$$

with

$$e_0=m,\quad \beta_i=\min\{j\mid a_j\neq 0,\ j\notin(e_{i-1})\},\quad e_i=\gcd(e_{i-1},\beta_i)\quad (i=1,\dots,g),$$

where m is chosen such that $e_g=1$. Since $e_i|e_{i-1}$, we can define $n_i=e_{i-1}/e_i\geqslant 2$.

These numerical invariants have a geometric meaning: e_0 is the multiplicity of f at the origin and e_i $(i=1,\dots,g)$ is the multiplicity of f at the i-th rupture divisor of its minimal embedded resolution, or equivalently the last infinitely near point of the i-th cluster of consecutive satellite points. These concepts are explained in [2].

From the Puiseux series we can define:

Definition 2.2. The *characteristic exponents* of a plane branch f are the rational numbers $\left(\frac{\beta_1}{m},\dots,\frac{\beta_g}{m}\right)$.

Following the notation of M. Saito [6] with a slight modification, let

$$\frac{\beta_i}{m}=1+\frac{l_1}{n_1}+\cdots+\frac{l_i}{n_1\cdots n_i}\quad (i=1,\dots,g)$$

with $n_j\geqslant 2$, $l_j\geqslant 1$, $\gcd(l_j,n_j)=1$. From this we define:

Definition 2.3. The *Puiseux pairs* of an irreducible plane curve f are $(n_1,l_1),\dots,(n_g,l_g)$.

The characteristic exponents and the Puiseux pairs are two equivalent sets of complete topological invariants of the singularity of f. That is: they determine, and are determined by, the homeomorphism class of $f^{-1}(0)\cap U$ for a small enough neighbourhood U of the origin.

Remark 2.4. The name Puiseux pairs appear in various slightly different ways in the literature. We based our definition on the one given by M. Saito [6], who used this name for the pairs $(k_1,n_1),\dots,(k_g,n_g)$ with $k_1=n_1+l_1$, $k_i=l_i$. Casas-Alvero [2] used the similar term characteristic pairs to refer to $(\beta_1,m),\dots,(\beta_g,m)$.

M. Saito [6] considered the following numerical invariants in order to obtain a formula for the characteristic function of the Hodge spectral exponents of an irreducible plane curve. We simplify the definition by letting $n_0 = 1$.

Definition 2.5. We define the following numerical invariants:

$$w_0 = 1, \quad w_j = n_j n_{j-1} w_{j-1} + l_j \qquad (j = 1, \dots, g),$$
$$\mu_0 = 0, \quad \mu_j = (n_j - 1)(w_j - 1) + n_j \mu_{j-1} \qquad (j = 1, \dots, g).$$

Proposition 2.6 ([4]). *The Milnor number of f is $\mu = \mu_g$. More generally, the Milnor number of a curve with Puiseux pairs $(n_1, l_1), \dots, (n_j, l_j)$ is μ_j, for any $j \in \{1, \dots, g\}$.*

From these definitions we prove:

Lemma 2.7. *The Milnor number of an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$ is*

$$\mu = \sum_{j=1}^{g} l_j e_j (e_{j-1} - 1) + (e_0 - 1)^2.$$

Thành and Steenbrink [8] already described the Hodge spectrum of any plane curve in terms of its topological invariants, but in this work we use a closed formula given by M. Saito:

Theorem 2.8 ([6, Theorem 1.5]). *The Hodge spectral exponents in the interval $(0, 1)$ are:*

$$\left\{ \frac{1}{e_j}\left(\frac{b}{n_j} + \frac{a}{w_j}\right) + \frac{c}{e_j} \;\middle|\; \begin{array}{l} 0 < a < w_j \\ 0 < b < n_j \\ 0 \leqslant c < n_{j+1} \cdots n_g \\ 1 \leqslant j \leqslant g \end{array} , \frac{b}{n_j} + \frac{a}{w_j} < 1 \right\}.$$

Notice that this formula gives us a set of $\mu/2$ Hodge spectral exponents and thus, by symmetry, it characterizes all the Hodge spectral exponents of f.

To work with characteristic functions (i.e., Fourier transforms), M. Saito defined:

Definition 2.9. Let $F(T) = \sum_{i \geqslant 0} a_i T^{i/N} \in \mathbb{C}[T^{1/N}]$. Then, we define the following truncations:

$$F^{<1}(T) = \sum_{i/N<1} a_i T^{i/N}, \quad F^{>1}(T) = \sum_{i/N>1} a_i T^{i/N},$$

which are the terms of $F(T)$ with exponents smaller and larger than 1 respectively.

Definition 2.10. We define the auxiliary functions $\Phi_j(T)$ as:

$$\Phi_1(T) = \frac{T^{1/w_1} - T}{1 - T^{1/w_1}} \frac{T^{1/n_1} - T}{1 - T^{1/n_1}},$$

$$\Phi_j(T) = \frac{1 - T}{1 - T^{1/n_j}} \Phi_{j-1}^{<1}(T^{1/n_j}) + T^{1-1/n_j} \frac{1 - T}{1 - T^{1/n_j}} \Phi_{j-1}^{>1}(T^{1/n_j}) + \frac{T^{1/w_j} - T}{1 - T^{1/w_j}} \frac{T^{1/n_j} - T}{1 - T^{1/n_j}}$$

(for $j = 2, \dots, g$).

Then, M. Saito proves the following theorem:

Theorem 2.11 ([6, Theorem 1.5]). *The characteristic function of an irreducible plane curve f is*

$$\chi_f(T) = \frac{1}{\mu_g} \Phi_g(T).$$

3. Limit distribution in the case of branches

In this section we study the case of plane branches for K. Saito's Question 1 on the limit distribution of Hodge spectral exponents. We consider irreducible plane curves $f \in \mathbb{C}\{x, y\}$ with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. In this case, K. Saito's Question 1 asks for which limits of irreducible plane curves f does the distribution of Hodge spectral exponents $D_f(s)$ converge to $N_2(s)$ (recall Definition 1.2). Equivalently, it asks for which limits of the Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$ does the characteristic function $\chi_f(T) = \mathcal{F}\{D_f(s)\}(\tau)$ converge to the Fourier transform $\mathcal{F}\{N_2(s)\}(\tau) = \left(\frac{T-1}{\log T}\right)^2$.

K. Saito computed the particular case of a limit where the invariant n_g of the last Puiseux pair tends to infinity while all the remaining Puiseux pairs are kept fixed. His result, recalling Proposition 1.6, is that the resulting limit distribution of Hodge spectral exponents is

$$\lim_{n_g \to +\infty} \chi_f(T) = \left(\frac{T-1}{\log T}\right)^2 = \mathcal{F}\{N_2\}(\tau).$$

This is the expected limit distribution of Question 1. Given this result, we are led to ask whether it is possible to generalize it for the following limits:

(i) $n_k \to +\infty$ for a particular $k \in \{1, \dots, g\}$ while keeping all other n_j and l_j fixed,

(ii) $l_k \to +\infty$ for a particular $k \in \{1, \dots, g\}$ while keeping all other n_j and l_j fixed.

For the first case we prove the following:

Theorem 3.1. *Let $g \in \mathbb{Z}_{>0}$, $k \in \{1, \dots, g\}$. Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Consider a sequence of such curves f with $n_k \to +\infty$, n_j $(j \neq k)$ fixed and all l_j fixed. Then, the limit of the characteristic function is*

$$\lim_{n_k \to +\infty} \chi_f(T) = \left(\frac{T-1}{\log T}\right)^2.$$

Equivalently, the limit of the distribution of Hodge spectral exponents is

$$\lim_{n_k \to +\infty} D_f(s) = N_2(s).$$

The preceding theorem states that sequences of irreducible plane curves with $n_k \to +\infty$ (with the other numerical invariants fixed) form a family of solutions to K. Saito's Question 1 on the limit distribution of Hodge spectral exponents.

On the other hand, we prove:

Theorem 3.2. *Let $g \in \mathbb{Z}_{>0}$, $k \in \{1, \dots, g\}$. Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Consider a sequence of such curves f with $l_k \to +\infty$, l_j $(j \neq k)$ fixed and all n_j fixed. Then, the limit of the characteristic function is*

$$\lim_{l_k \to +\infty} \chi_f(T) = \frac{1}{e_{k-1} - 1} \frac{T-1}{\log T} \frac{T^{1/e_{k-1}} - T}{1 - T^{1/e_{k-1}}}.$$

Equivalently, the limit of the distribution of Hodge spectral exponents is

$$\lim_{l_k \to +\infty} D_f(s) = \frac{1}{e_{k-1} - 1}(\lfloor e_{k-1} s \rfloor \, \mathbb{1}_{[0,1)}(s) + \lfloor e_{k-1}(2 - s) \rfloor \, \mathbb{1}_{[1,2)}(s)),$$

where $\mathbb{1}_{[a,b)}(s)$ denotes the indicator function of the interval $[a, b)$.

These limits are different from $\left(\frac{T-1}{\log T}\right)^2$ and $N_2(s)$ respectively. Therefore, this theorem says that sequences of irreducible plane curves with $l_k \to +\infty$ (with the other numerical invariants fixed) are a family of *non*-solutions to K. Saito's Question 1 on the limit distribution of Hodge spectral exponents.

4. Cumulative difference function ϕ_f

From Definition 1.3 we have that the cumulative difference function for the case of plane curves is $\phi_f\colon [0,1) \to \mathbb{R}$ defined as

$$\phi_f(r) = \int_0^r N_2(s) - D_f(s)\,\mathrm{d}s = \int_0^r N_2(s) - \frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)\,\mathrm{d}s = \frac{1}{2}r^2 - \frac{1}{\mu}\#\{\alpha_i \leqslant r\}$$

since we have $N_2(s) = s$ in the interval $[0,1)$.The Hodge spectral exponents α_i are given by Theorem 2.8.

To be able to study the set of dominating values (i.e., K. Saito's Question 2) we need a more explicit expression for $\#\{\alpha_i \leqslant r\}$ or equivalently $\phi_f(r)$. For this purpose we prove the following:

Theorem 4.1. *Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Then, for any $r \in [0,1)$, the number of Hodge spectral exponents less or equal to r is given by the following expression:*

$$\#\{\alpha_i \leqslant r\} = \frac{\mu_g - n_g w_g}{2}r + \frac{n_g w_g}{2}r^2 + \sum_{j=1}^{g}\frac{n_j - 1}{2}\{e_j r\} + \frac{1}{2}\{e_0 r\}(1-\{e_0 r\})$$
$$+ \sum_{j=1}^{g}\frac{l_j}{2n_j}\{e_{j-1}r\}(1-\{e_{j-1}r\}) - \sum_{j=1}^{g}\sum_{b=1}^{n_j-1}\left\{w_j\left(\{e_j r\} - \frac{b}{n_j}\right)\right\}\mathbb{1}_{\left[\frac{b}{n_j},1\right)}(\{e_j r\}).$$

Theorem 4.2. *Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Then, for any $r \in [0,1)$, the cumulative difference function between $N_2(s)$ and $D_f(s)$ is given by the following expression:*

$$\phi_f(r) = \frac{1}{2\mu}\left(\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)r(1-r) - \sum_{j=1}^{g}(n_j-1)\{e_j r\} - \{e_0 r\}(1-\{e_0 r\})\right.$$
$$\left. - \sum_{j=1}^{g}\frac{l_j}{n_j}\{e_{j-1}r\}(1-\{e_{j-1}r\}) + \sum_{j=1}^{g}\sum_{b=1}^{n_j-1} 2\left\{w_j\left(\{e_j r\} - \frac{b}{n_j}\right)\right\}\mathbb{1}_{\left[\frac{b}{n_j},1\right)}(\{e_j r\})\right).$$

5. Dominating values for irreducible plane curves

In this section we give partial answers to K. Saito's Question 2 on the dominating values for the case of irreducible plane curves. To such purpose we use Theorem 4.2, which gives us an explicit expression for the cumulative difference function $\phi_f(s)$. This expression can be used to find simpler functions which bound $\phi_f(s)$, thus making it possible to obtain intervals where ϕ_f is positive. We prove the following result:

Theorem 5.1. *Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Then,*

(i) A set of dominating values is given by the interval

$$r \in \left(\frac{(2e_0 - n_g + \sum_{j=1}^{g} l_j e_j) - \sqrt{D_1}}{2(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j)}, \frac{(2e_0 - n_g + \sum_{j=1}^{g} l_j e_j) + \sqrt{D_1}}{2(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j)} \right)$$

with

$$D_1 = \left(2e_0 - n_g + \sum_{j=1}^{g} l_j e_j\right)^2 - 4\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)\left(\sum_{j=1}^{g-1}(n_j - 1) + \frac{1}{4} + \sum_{j=1}^{g} \frac{l_j}{4n_j}\right) > 0.$$

(ii) A set of dominating values is given by the interval

$$r \in \left(\frac{(e_0 + \sum_{j=1}^{g} l_j e_j) - \sqrt{D_2}}{2(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j)}, \frac{(e_0 + \sum_{j=1}^{g} l_j e_j) + \sqrt{D_2}}{2(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j)} \right)$$

with

$$D_2 = \left(e_0 + \sum_{j=1}^{g} l_j e_j\right)^2 - 4\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)\left(\frac{1}{4} + \sum_{j=1}^{g} \frac{l_j}{4n_j}\right) > 0.$$

(iii) We have that the leftmost interval of $(0, 1)$

$$r \in (0, \operatorname{lct}(f)) = \left(0, \frac{1}{e_1}\left(\frac{1}{n_1} + \frac{1}{n_1 + l_1}\right)\right)$$

is a set of dominating values. In contrast, $\phi_f(r) < 0$ for the rightmost interval

$$r \in \left[1 - \frac{1}{n_g w_g}, 1\right).$$

We point out that the first two intervals always intersect but it is not always clear which are the ends of the unique interval of dominating values they provide. The intervals of the third item are obtained directly from the smallest and largest Hodge spectral exponents.

Remark 5.2. Almirón and Schulze [1, Proposition 6] proved that the log-canonical threshold of an irreducible plane curve is a dominating value except for the cases where the curve has semigroup $(2, 3)$ or $(2, 5)$.

In the course of the proof of Theorem 5.1 we also obtain an alternative proof of Theorem 1.8 by Tomari [9] but only for irreducible curves.

Acknowledgements

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Extended Abstracts

Societat Catalana de Matemàtiques

Institut d'Estudis Catalans

SCM Master Thesis Day

On October 4, with a large attendance, we celebrated the second SCM TFM Day. This activity, organized by the Catalan Mathematical Society (SCM), aims to provide recent graduates from a master's program in mathematics at a Catalan university or within the common linguistic area (Xarxa Vives) the opportunity to present their Final Master's Thesis. This inter-university event offers young graduates a platform to participate in their first workshop presentation, fostering connections within the community of early-career mathematicians as they embark on research. It also serves to inform participants about the Galois Awards, the journal *Reports@SCM*, and the master's programs in mathematics at universities in the Vives Network—information which is particularly relevant to mathematics students in their final year who attend the event.

The event was held at the headquarters of the Institut d'Estudis Catalans and featured eight students as speakers, along with presentations by the two students who won the Evariste Galois 2024 Prize: Pablo Nicolás (winner) and Roger Gómez (recipient). This award, given by the SCM for the best final master's thesis of the previous year (in this case, 2023), highlights outstanding research achievements. Notably, the two winners of the 2024 Galois Award had presented their TFM at the 2023 SCM TFM Day.

The scientific and organizing committee for the event included Montserrat Alsina (President of the SCM), Josep Vives (Vice-President of the SCM), Ainoa Murillo (SCM board member), Simone Marchesi (Editor of *Reports@SCM*), Xavier Massaneda (Coordinator of the Master of Advanced Mathematics UB-UAB), Jordi Saludes (Coordinator of the UPC Master's in Advanced Mathematics), Enric Cosme and Pablo Sevilla (Coordinators of the University Master's in Mathematical Research UV-UPV), along with Pablo Nicolás, Philip Pita, and Sergi Sánchez, current PhD students and participants in the 2023 edition.

This issue of *Reports@SCM* includes extended abstracts of the ten presentations given during the event.

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An introduction to stochastic integration

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Resum *(CAT)*
L'objectiu d'aquest treball és el d'estudiar integrals estocàstiques que no són necessàriament respecte del moviment brownià.
Primer de tot es revisa la construcció d'aquesta darrera integral per motivar les possibles extensions a altres integradors com són les martingales.
A continuació, estudiem les integrals respecte de camps aleatoris, on comencem per estudiar aquestes integrals respecte del soroll blanc gaussià per, un cop més, estendre la classe d'integradors.
Alhora que es van estudiant aquests objectes, també presentem alguns resultats referents a l'aproximació en llei d'aquests.

Keywords: *Brownian motion, Gaussian process, white noise, martingale, stochastic integral, convergence in law.*

Abstract

The main purpose of this work is to continue and extend the study of the stochastic integral with respect to the Brownian motion usually seen in courses of Stochastic Calculus, providing (hopefully) an introductory text that will allow the average student of the subject (and to anyone who is already familiar with the previously mentioned stochastic integral) to expand his knowledge.

To do so, we briefly review the construction of the Itô integral with respect to the Brownian motion and notice that it turns out that very few features of this particular process are used, which allows us to exploit these ideas to generalize the construction to other processes (mainly, martingales), this is done following the construction provided in the third chapter of [2].

We then discuss the topic of stochastic integration with respect to random fields. We first treat the integral with respect to the space-time Gaussian white noise, following the construction presented in the first two chapters of [1], since it deals with objects which might be a bit more familiar to the intended audience as its construction uses the already studied Itô integral with respect to the Brownian motion and Parseval's identity. Before doing so, we introduce two crucial Gaussian processes (the isonormal process and the white noise), which generalize the Brownian motion and are crucial when it comes to define the stochastic integral with respect to the space-time white noise.

Next, and following the second chapter of [3], we introduce a wider class of random fields (which contains the ones already seen) that can be used as integrators and show how one constructs integrals with respect to such objects. During this process, we use the already studied Gaussian white noise as a canonical example that will serve us as a model to compare the new construction.

Finally, and as we study these objects, we address the problem of how the integrals with respect to the Brownian motion and with respect to the space-time Gaussian white noise can be approximated in law by integrals with respect to random walks. The results obtained, which are motivated by the already known invariance principles like the Donsker's one, can be seen as generalizations of these since the latter can be obtained as a particular case of the former.

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Stochastic differential equations driven by a fractional Brownian motion

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Resum *(CAT)*
En aquest treball s'estudien les equacions diferencials estocàstiques (EDEs) dirigides per un moviment brownià fraccionari (fBm) amb paràmetre de Hurst $H > 1/2$. Es defineix la integral estocàstica respecte al fBm i es demostra l'existència i unicitat de solucions. També s'introdueix el càlcul de Malliavin en el context del fBm, i es prova que, amb condicions més fortes en els coeficients, la llei de la solució és absolutament contínua. Finalment, es donen fites d'estil gaussià per la densitat d'una família d'EDEs.

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Keywords: *fractional Brownian motion, stochastic differential equations, Malliavin calculus.*

Abstract

In the thesis, we study from several points of view the so-called stochastic differential equations driven by a fractional Brownian motion with Hurst index $H > 1/2$. These objects are differential equations of the form

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s^H + \int_0^t b(s, X_s)\,\mathrm{d}s, \quad t \in [0, T], \tag{1}$$

where B^H is a fractional Brownian motion with $H \in (1/2, 1)$, that is, a centered stochastic Gaussian process with covariance function

$$R_H(t, s) := E(B_t^H B_s^H) = \frac{t^{2H} + s^{2H} - |t - s|^{2H}}{2}.$$

Notice that, in particular, when $H = 1/2$ then B^H is a standard Brownian motion. The first topic covered in the thesis is giving sense to an equation like (1). The fact that B^H is not a semimartingale makes it impossible to define the integral with respect to B^H in a similar way as it is defined for the standard Brownian motion. In virtue of the results of Young in [4] and the further contributions of Zähle in [5], we are able to define the integral with respect to B^H in the generalized Stieltjes sense. Once the stochastic integral is well-defined, we follow closely the arguments of Nualart and Răşcanu in [2] to prove the existence and uniqueness of solutions to a general SDE of the form (1).

Once we know we can talk about the solution to equation (1), then we want to study such solution from a probabilistic point of view. Using the Malliavin calculus (that is, the stochastic calculus of variations)

we get to prove that under stronger hypothesis on the coefficients σ and b, the law of X_t is absolutely continuous with respect to the Lebesgue measure, so for each $t \in [0, T]$ X_t has a density function $p_t(x)$. In order to prove this result, we use the concepts of Malliavin differentiability in the fractional Brownian motion framework and Fréchet differentiability and we use the same techniques as in Nualart and Saussereau in [3].

Finally, using more sophisticated Malliavin calculus techniques and the method of Nourdin–Viens in [1] we are able to proof that the solution X_t to an equation of the type

$$X_t = x_0 + \int_0^t \sigma_s \, \mathrm{d}B_s^H + \int_0^t b(s, X_s) \, \mathrm{d}s, \tag{2}$$

where σ is deterministic, σ and b satisfy the same hypothesis as for the existence of the density function $p_t(x)$ and we assume further that there exist $0 < \lambda < \Lambda$ such that $\lambda < \sigma_s < \Lambda$, then the density $p_t(x)$ for $t \in (0, T]$ is bounded in the following way:

$$\frac{E(|X_t - m_t|)}{2\Lambda^2 t^{2H}} \exp\left(-\frac{(x-m_t)^2}{2\lambda^2 t^{2H}}\right) \leq p_t(x) \leq \frac{E(|X_t - m_t|)}{2\lambda^2 t^{2H}} \exp\left(-\frac{(x-m_t)^2}{2\Lambda^2 t^{2H}}\right),$$

where $m_t = E(X_t)$.

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Regularity of Lipschitz free boundaries in the Alt–Caffarelli problem

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Resum *(CAT)*
En aquest treball estudiem la regularitat de les fronteres lliures Lipschitz en el problema d'Alt–Caffarelli. Demostrem que les fronteres lliures Lipschitz són $C^{1,\alpha}$ mitjançant la invariància per reescalament del problema i la regularitat inicial Lipschitz de la frontera. A més a més, també provem que les fronteres $C^{1,\alpha}$ són C^∞, cosa que, juntament amb el resultat anterior, implica que les fronteres lliures Lipschitz són C^∞.

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Keywords: *partial differential equations, free boundary problems, Alt–Caffarelli problem, Lipschitz regularity.*

Abstract

Free boundary problems are a subclass of PDEs in which not only do we have to solve a particular PDE, but we also have to find an unknown domain Ω where our solution solves the problem. More precisely, we have a fixed (smooth) domain D and we want to find a pair (Ω, u) such that $\Omega \subset D$ is a domain and $u\colon \Omega \to \mathbb{R}$ is a solution in Ω of the PDE in question. The term *free boundary* refers to $\partial\Omega \cap D$, that is, the piece of the boundary of Ω that falls inside D. The word *free* signifies the fact that the free boundary depends on our solution and will change as soon as our solution does so.

Motivated by its relevance in fields such as fluid mechanics, optimal design problems and electrostatics, the Alt–Caffarelli problem (sometimes also called the one-phase problem or the Bernoulli problem) is a classical example of a free boundary problem. Studied for the first time in [1], this problem consists in finding a nonnegative function u defined in $B_1 = B_1(0)$ solving

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap B_1, \\ \partial_\nu u = 1 & \text{on } \partial\{u > 0\} \cap B_1. \end{cases} \tag{1}$$

In the general notation used previously, our fixed domain is $D = B_1$, $\Omega = \{u > 0\}$, and the free boundary is $\partial\{u > 0\} \cap B_1$. Observe that in (1) we are imposing two boundary conditions on the free boundary: $u = 0$ (implicitly) and $\partial_\nu u = 1$. This type of problem will not have a solution in general since it constitutes an overdetermined PDE problem. However, if the problem can be solved (which is the case for the Alt–Caffarelli problem), then we may expect to prove extra properties of the free boundary $\partial\{u > 0\} \cap B_1$ thanks to the overdetermination of the problem.

In this work we study the Lipschitz regularity of the free boundary $\partial\{u > 0\} \cap B_1$. More precisely, we assume the free boundary to be Lipschitz and then show how to improve its regularity by exploiting the overdetermined nature of the problem. The main result we focus on is the following one:

Theorem. *Let u be a (viscosity) solution of the Alt–Caffarelli problem (1). Assume that the free boundary $\partial\{u > 0\} \cap B_1$ is Lipschitz. Then $\partial\{u > 0\} \cap U$ is smooth for any open set $U \Subset B_1$. Moreover, $u \in C^\infty(\overline{\{u > 0\}} \cap U)$ and u solves (1) in the classical sense in U.*

The proof of this theorem is accomplished in two steps: first, by proving that Lipschitz free boundaries are $C^{1,\alpha}$, and second, by showing that $C^{1,\alpha}$ free boundaries are smooth. For the first step, the main idea is to use the rescaling invariance of (1). Notice that if u is a solution, then for any $r > 0$ the function $u_r(x) = \frac{1}{r}u(rx)$ is also a solution in the corresponding rescaled domain $B_{1/r}$. This property is crucial to finish this first step. Geometrically, the Lipschitz regularity of $\partial\{u > 0\} \cap B_1$ implies that the free boundary always remains outside a cone of a fixed opening. Using that our solution satisfies (1), we are able to show that we can improve the opening of this cone in the ball $B_{1/2}$. However, this alone is clearly insufficient to conclude that the free boundary is $C^{1,\alpha}$. What enables us to complete the proof of this first step is precisely the rescaling invariance of the problem, which we use to repeat the opening improvement iteratively in the sequence of balls $B_{2^{-k}}$. Intuitively, this process tells us that the free boundary "flattens" as we zoom in at the origin which implies, after some extra steps, that the free boundary is $C^{1,\alpha}$.

As for the second step, we perform some computations combined with Schauder estimates for the Laplacian to show that once we have $C^{1,\alpha}$ regularity on the free boundary, then we can improve the boundary as much as we want to obtain C^∞ regularity. Lastly, combining both steps and using a simple covering argument we obtain the proof of the theorem.

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Spectral gap of generalized MIT bag models

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Resum *(CAT)*
Estudiem propietats espectrals dels models de bossa de l'MIT generalitzats. Aquests són operadors de Dirac $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ actuant en dominis de $\mathbb{R}^3$ amb condicions de frontera que generen confinament. El seu autovalor positiu més baix és d'especial interès, i s'ha conjecturat que és mínim per a una bola entre tots els dominis amb volum fixat. La conjectura anàloga és certa per al laplacià de Dirichlet (és la desigualtat de Faber–Krahn), que sorgeix en els límits $\tau \to \pm\infty$. Estudiant la convergència en el sentit de la resolvent dels operadors $\mathcal{H}_\tau$ cap als operadors límit $\mathcal{H}_{\pm\infty}$ quan $\tau \to \pm\infty$, provem que certes propietats espectrals s'hereden al llarg de la parametrització. Aquests resultats són nous i s'han publicat a [3].

Keywords: *Dirac operator, spectral theory, MIT bag model, shape optimization, resolvent convergence.*

Abstract

The equation that governs all relativistic quantum processes is called Dirac equation. In $\mathbb{R}^3$, it is a complex valued system of four linear PDEs of first order in time and space variables. For a spin-$1/2$ free particle of mass m, one can write the Dirac equation in matricial form as $i\frac{\partial}{\partial t}\psi(x,t) = (-i\alpha\cdot\nabla + m\beta)\psi(x,t)$, where α and β are the so-called Dirac matrices, given —essentially— by the more known Pauli matrices. The stationary eigenvalue problem associated to the Dirac equation is of the form

$$\begin{cases} (-i\alpha\cdot\nabla + m\beta)\,\varphi = \lambda\varphi & \text{in } \Omega, \\ \text{Boundary conditions} & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is the domain where the particle evolves, $\varphi\colon \Omega \to \mathbb{C}^4$, and the boundary conditions typically depend on physical constraints. The eigenvalues λ of such Dirac operators provide relevant information to understand the evolution of the system, and hence their study is of special interest.

Dirac operators acting on domains $\Omega \subset \mathbb{R}^3$ are used in relativistic quantum mechanics to describe particles that are confined in a box. The so-called *MIT bag model* is a very remarkable example, which was introduced in the 1970s as a simplified model to study confinement of quarks in hadrons.

In [1] it is introduced a family $\{\mathcal{H}_\tau\}_{\tau\in\mathbb{R}}$ of Dirac operators with confining boundary conditions parameterized by $\tau \in \mathbb{R}$; the particular case $\tau = 0$ corresponds to the MIT bag model. Because of this reason, the operators $\mathcal{H}_\tau$ are called *generalized MIT bag models*.

In this work [2], we study some spectral properties of generalized MIT bag models. Their lowest positive eigenvalue is of special interest, and it is conjectured to be minimal for a ball among all domains of the same volume. The analogous conjecture holds true for the Dirichlet Laplacian (it is the Faber–Krahn inequality).

We prove that the Dirichlet Laplacian arises in the limit $\tau \to \pm\infty$ by studying the resolvent convergence of $\mathcal{H}_\tau$ in this limit. More specifically, we show strong resolvent convergence of $\mathcal{H}_\tau$ to $\mathcal{H}_{\pm\infty}$, and we justify that one cannot improve this to norm resolvent convergence. These results are new and have been published in [3], together with other extended results.

Because of this convergence, we show that some spectral properties of the limiting operators $\mathcal{H}_{\pm\infty}$ are inherited throughout the parameterization. As a consequence, we verify the conjecture for large enough values of the parameter τ.

Finally, we prove that the conjecture holds true for corona domains of relatively small hole. This result is also new. However, a continuation of this study after the master's thesis —using more abstract arguments— allowed to complete this result for any corona of the same volume (independently of the size of its hole). This extended result will be sent for publication in an indexed journal.

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Unifying framework for decision-making dynamics: optimal control and infinite horizon perspectives

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Resum *(CAT)*
L'objectiu principal d'aquest projecte és desenvolupar un marc teòric unificat de la propriocepció, el control motor i la presa de decisions. Primerament, es presenta una introducció al càlcul variacional i la teoria del control òptim per establir una base teòrica sòlida. A continuació, es proposa un sistema dinàmic lineal com a aproximació al sistema físic estudiat i es desenvolupa un model seqüencial per predir trajectòries. Davant la dificultat de trobar una solució analítica exacta, s'utilitza el filtre de Kalman per estimar els perfils de posició i velocitat.

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Keywords: *motor control, decision-making, proprioception, optimal feedback control (OFC).*

Abstract

Movement is the only way to express our thoughts and moods, which in its full expression determines our overall behaviour. A vast amount of research has been devoted to study how the brain generates and controls movement over the last century. Furthermore, the study of the principles underlying how the brain generates movement are of significant relevance both from a scientific but also clinical perspective, as most disorders are often quantified in terms of the motor deficits they imply, e.g., Parkinson's disease, ictus or simple ageing.

Recent studies have described the generation and control of movements in terms of the benefits and costs associated with potential movements [7], thus establishing a fundamental relationship between movement generation and decision-making theory. The combination of movement related choices and more cognitive decisions determines our responses and the behaviour with which we interact with the environment. This can be studied and modelled mathematically through *optimal decision making* and *motor control theory* [8]. However, these theories fall short to consider the contribution and role of the inner perception of our body, namely the bodily perception or *proprioception*, which plays a crucial role when planning and executing movements. In particular, proprioception provides internal corroboration that a movement is ongoing, it is hence a distributed phenomenon implicated in processes of *top-down prediction* and *bottom-up correction*.

Despite its obvious practical and clinical importance, proprioception remains one of the least studied senses, often overshadowed by its more familiar counterparts. Consequently, the central purpose of this project is to present a *unified theory* that, unlike simpler models of motor control, encompasses the *explicit incorporation of neural signatures of proprioception* into a *comprehensive model*. The proposed model may be able to describe the *interactions between proprioception and its influence on motor control*.

For this purpose, on the first chapter, an introduction to the state of the art theories and recent work on the principles of optimal feedback control (OFC) and movement related choices is presented, as stated by experts of the field, [6, 8], to fully contextualize the problem.

On the second chapter, robust theoretical formulation on optimal control framework is presented in order to provide a solid mathematical background as a means to understand the dynamical system studied as an Infinite horizon optimal control problem, [2, 3, 5].

The experimental approach followed on this project is based on planar reaching movements, as described on previous studies [4]. On Chapter 3, an introduction to time series theory is presented to understand the data analysis conducted over the experimental distributions gathered, since they are given by time series to apply the optimal feedback control approach as described by the Kalman filter, [1]. On Chapter 4, a more detailed insight to the experimental configuration is stated, concerning the explicit development of the dynamical system studied, as proposed by [6, 7]. Following the theoretical framework built in the previous chapters, the sequential computational approach implemented is stated on Chapter 5.

Finally, the results obtained from implementing the OFC model are stated in Chapter 6, as well as an exhaustive comparison and analysis between the experimental and simulated distributions.

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Throughout the writing of this thesis, I have received a great deal of support and assistance from my supervisor, Dr. Ignasi Cos Aguilera.

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Moduli of plane branches with a single characteristic exponent

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Resum *(CAT)*
Estudiem l'espai de mòduls de branques planes (per equivalència analítica) amb un únic exponent característic mitjançant una estratificació utilitzant el semigrup de valors de l'ideal jacobià de la branca. En particular, estudiem com abordar el problema mitjançant diferents tècniques. En primer lloc, proporcionem un procediment algorítmic basat en un procediment de Casas-Alvero, que, sota alguns supòsits, descriu els estrats. Incloem una implementació en `Maple` d'aquest algorisme. A més, comparem la nostra estratificació amb una altra estudiada prèviament per Peraire l'any 1998 a partir de l'invariant de Zariski. Això ens permet fer algunes reflexions sobre els reptes de calcular la dimensió dels nostres estrats, que refinen els estrats de Peraire, i presentar algunes noves eines per abordar el problema.

Keywords: *analytic classification, stratification, Jacobian ideal.*

Abstract

The moduli problem of classifying by analytical equivalence germs of irreducible curves in the same equisingularity class was first introduced by Oscar Zariski, [3], who gave a partial description of the space and worked out some examples. The difficulty of the problem soon became apparent and many open questions, that remain still open, arose. Our goal is to study the moduli problem for plane branches with a single characteristic exponent and describe a stratification using the semigroup of values of the Jacobian ideal of the branch, denoted by Θ. This stratification refines a previously known one based on the Zariski invariant studied by Rosa Peraire in 1998, [2].

The germs

$$\gamma_{\mathbf{A},\sigma} : f_{\mathbf{A},\sigma} = y^n - x^m + x^p y^q + \sum_{\substack{ni+mj>nm+\sigma \\ 0\le i\le m-2 \\ 0\le j\le n-2}} A_{i,j} x^i y^j = 0, \quad A_{i,j} \in \mathbb{C},$$

where we denote by σ the integer $np + mq - nm$ and by $\mathbf{A}$ the coefficients $A_{i,j}$, represent all analytic types of germs with a single characteristic exponent equal to m/n, $(n, m) = 1$, and Zariski invariant equal to σ. Casas-Alvero in [1] describes a procedure that obtains for a given $f(\mathbf{A}, x, y) \in \mathbb{C}(\mathbf{A})\{x, y\}$, as in the previous equation with fixed Zariski invariant σ, the semigroup of values of the Jacobian ideal of the branch $\gamma_{\mathbf{A}}$ according to a set of conditions on the $A_{i,j}$. These conditions describe the strata that correspond to the Zariski invariant σ of the stratification of the moduli space using the semimodule of values of the

Jacobian ideal of a branch. One of our main contributions is developing this procedure into an algorithm and implementing it in `Maple`. The implementation of this procedure also lead us to the construction of an interesting tree, the tree of conditions for the Jacobian values of a fixed single characteristic exponent m/n. A rooted tree structure in which the leaves represent all the possible semigroups of values of the Jacobian ideal of $\gamma_{\mathbf{A}}$ and the nodes represent the conditions that the set of coefficients must satisfy in each of the cases.

From our algorithm we are able to deduce a semi-reduced and a reduced equation describing of all the branches in any fixed stratum with fixed set of Jacobian values Θ, and as a by-product the dimension of that stratum. Given any stratification, a general equation $f_{\mathbf{A},\sigma}$ of a stratum (representing all its analytic types) is semi-reduced if the only non-null coefficients $A_{i,j}$ are those whose variation results in a change in the analytical type of the branch $\gamma_{\mathbf{A},\sigma}$. A reduced equation of a stratum is a semi-reduced equation expressed in terms of a minimal number of $A_{i,j}$. These $A_{i,j}$ in a reduced equation provide a parametrization of the stratum, and its cardinal is precisely the dimension of that stratum.

Peraire in her Theorem 4.12 of [2] gives a combinatorial expression for the dimension of the strata of her stratification by the Zariski invariant. We generalize her result and prove that an analogous expression accounts for the number of non-null coefficients $A_{i,j}$ in a semi-reduced equation. Furthermore, we prove that for her strata, any semi-reduced equation is, in fact, reduced, which does not hold in general for our strata. We reveal that this presents the main difficulty in providing a combinatorial expression for the dimension of the strata in our stratification by the set of Jacobian values Θ. Finally we introduce the notion of Θ-continuous coefficients, which we believe is strongly related to this dimension.

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Geometric methods in monogenic extensions

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Resum *(CAT)*
Un cos de nombres K és monogen si el seu anell d'enters està generat per un sol element com a $\mathbb{Z}$-àlgebra. En el cas cúbic, determinar si K és monogen o no és equivalent a resoldre l'equació diofàntica $|I_K(X,Y)| = 1$ sobre $\mathbb{Z}$, on I_K és la forma índex del cos. Una solució entera determina un punt racional a la corba de gènere 1 $I_K(X,Y) = Z^3$. Mitjançant aquesta construcció, es pot demostrar que K determina una $\mathbb{F}_3$-òrbita en $H^1(\mathbb{Q}, E[3])$, on E és la corba el·líptica definida per $Y^2 = 4X^3 + \mathrm{Disc}(K)$. Donem la construcció explícita d'aquesta òrbita pel cas de cossos cúbics purs i caracteritzem la suma de cocicles associats a cossos no isomorfs.

Keywords: *monogenity, diophantine equations, elliptic curves, Galois cohomology.*

Abstract

It is well known, due to the primitive element theorem, that any number field K is generated by a single algebraic number over $\mathbb{Q}$. One would think that the analogous statement should hold for the ring of integers $\mathcal{O}_K$, so that $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some algebraic integer $\alpha \in \mathcal{O}_K$. However, Dedekind found the first counterexample for this assumption in 1878 (see [2]). When there exists such α, K is said to be *monogenic*. Today, for $n \geq 3$, it is expected that, when ordered by discriminant, the set of monogenic number fields of degree n has measure 0 (see [1]).

After choosing a suitable integral basis of $\mathcal{O}_K$, we can associate a degree $n(n-1)/2$ homogeneous form I_K on $n-1$ variables to K called the *index form* of K. This form allows to characterize the monogenity of K by a diophantine equation: K is monogenic if, and only if, there exist $x_1, \dots, x_{n-1} \in \mathbb{Z}$ such that $I_K(x_1, \dots, x_{n-1}) = \pm 1$. When $n = 3$, I_K is a binary cubic homogeneous form and its discriminant is equal to the discriminant of K (see [3]). Thus, the projective curve $C_K : I_K(X,Y) = Z^3$ is smooth and an integral solution to the index form equation gives rise to a rational point on C_K. Therefore, we focus on studying the existence of rational points on C_K.

For non-zero $r \in \mathbb{Q}$, let E^r denote the elliptic curve given by $Y^2 = 4X^3 + r$. Let $D = \mathrm{Disc}(K)$. In recent work of Alpöge, Bhargava, and Shnidman (see [1]), they defined a rational map $\pi_K \colon C_K \to E^{-27D}$ and a 3-isogeny $\phi_D \colon E^D \to E^{-27D}$ such that (C_K, π_K) is a ϕ_D-covering of E^{-27D}. As a consequence, C_K is a homogeneous space for E^D. The ϕ_D-coverings of E^{-27D} are parametrized by $H^1(\mathbb{Q}, E^D[\phi_D])$,

where $E^D[\phi_D] = \ker(\phi_D)$, and homogeneous spaces for E^D are parametrized by the Weil–Châtelet group $H^1(\mathbb{Q}, E^D)$, whose trivial class consists of the homogeneous spaces for E^D which have a rational point. These cohomology groups are related by the Kummer exact sequence

$$E^{-27D}(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, E^D[\phi_D]) \xrightarrow{\iota} H^1(\mathbb{Q}, E^D),$$

where ι is given, in terms of ϕ_D-coverings and homogeneous spaces, by $\iota(C_K, \pi_K) = C_K$. Thus, (C_K, π_K) is in the kernel of ι if, and only if, C_K has a rational point. Therefore, by analysing this kernel, we can study the monogenity of families of number fields with discriminant D. We apply this theory in order to give bounds for the total number of monogenic cubic number fields with the same discriminant in terms of E^D.

Since (C_K, π_K) is a ϕ_D-covering, it determines a class $\alpha_K \in H^1(\mathbb{Q}, E^D[\phi_D])$. When K is a Dedekind type I field, i.e. when $K = \mathbb{Q}(\sqrt[3]{hk^2})$, where h, k are coprime, square-free integers such that $hk^2 \not\equiv \pm 1$ modulo 9, we prove that the cocycle

$$\xi_K : \sigma \longmapsto \log_\omega \left(\frac{\sigma(\sqrt[3]{hk^2})}{\sqrt[3]{hk^2}} \right) (0 : \sqrt{D} : 1)$$

is a representative of α_K, where ω is a third root of unity. Using this expression, given Dedekind type I fields K_1 and K_2 with the same discriminant, we find an expression for the ϕ_D-covering associated to $\xi_{K_1} + \xi_{K_2}$, determining also when this covering corresponds to a Dedekind type I field.

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On the basins of attraction of root-finding algorithms

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Resum *(CAT)*
Els algoritmes de cerca d'arrels han estat històricament utilitzats per resoldre numèricament equacions no lineals de la forma $f(x) = 0$. Aquest treball explora la dinàmica dels mètodes de la família Traub parametritzada $T_{p,\delta}$ aplicada a polinomis. Aquests mètodes inclouen un ventall des del mètode de Newton ($\delta = 0$) fins al mètode de Traub ($\delta = 1$). El nostre enfocament rau a investigar diverses propietats topològiques de les conques d'atracció, particularment la seva simple connectivitat i la no acotació, que són crucials per identificar un conjunt universal de condicions inicials que assegurin la convergència a totes les arrels de p.

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Keywords: *dynamical systems, root-finding algorithms, Newton's method, Traub's method.*

Abstract

Solving nonlinear equations of the form $f(x) = 0$ is a common challenge in various scientific fields, spanning from biology to engineering. When algebraic manipulation is not feasible, iterative methods become necessary to determine solutions. Newton's method is a well-known approach, derived from linearizing the equation $f(x) = 0$. Its iterative expression is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

Over the past few decades, numerous researchers have suggested various iterative approaches aimed at enhancing Newton's method [4]. One prevalent strategy for devising new methods involves directly combining existing techniques and subsequently modifying them to minimize the count of functional evaluations. For example, if we apply Newton's method twice while keeping the derivative constant in the second step, we derive Traub's method [5]. A specific type of root-finding algorithms, called the *damped Traub's family*, was first introduced in the papers [2, 6]. Its iterative expression is given by:

$$x_{n+1} = y_n - \delta \frac{f(y_n)}{f'(x_n)}, \quad n \geq 0,$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ is a Newton's step and δ is the damping parameter. Notice that $\delta = 0$ corresponds to Newton's method and $\delta = 1$ to Traub's method. Newton's method converges quadratically for simple

roots of a polynomial when the initial guess is sufficiently close to the desired root. On the other hand, Traub's method exhibits cubic (local) convergence. It is worth noting that each iteration of Traub's method requires more computations compared to Newton's method.

The challenge of iterative methods lies in the choice of initial conditions to start the algorithm. The study of dynamical systems plays a crucial role in gaining insight into how to make this selection effectively. An example of this is presented in [3], where a universal and explicit set of initial conditions, denoted by $\mathcal{S}_d$, is constructed. This set depends solely on the degree of the polynomial and can be used to find all the roots of a polynomial using Newton's method. The existence of this set is ensured by the fact that the immediate basins of attraction for the method are simply connected and unbounded sets.

The aim of this work is to construct a set analogous to $\mathcal{S}_d$ for Traub's method. If successful, this would offer a way to find all the roots of a polynomial with enhanced convergence speed. To achieve this, it is necessary to prove that the immediate basins of attraction for Traub's method are simply connected and unbounded sets. This would provide the essential framework for constructing a set similar to $\mathcal{S}_d$. A recent study [1] demonstrated this result under the assumptions that the polynomial is either of degree 2, or it can be expressed in the form $p_{n,\beta}(z) := z^n - \beta$, where $n \geq 3$ and $\beta \in \mathbb{C}$.

We contributed to this research by analyzing the behavior of the damped Traub's family when the damping factor is close enough to zero by considering the method as a singular perturbation. We have been successful in proving the unbounded nature of the immediate basins of attractions for this case. Furthermore, we focus on investigating the simple connectivity and unboundedness of the immediate basins of attractions specifically for third-degree polynomials, achieving some findings concerning the distribution of both the free critical points and the fixed points that are not roots for the damped Traub's method under the condition that δ is close to zero. Finally, we conclude our research by examining Traub's method applied to the family $p_d(z) = z(z^d - 1)$. We have proven the unboundedness of the immediate basins of attraction for specific values of d, and we present evidences suggesting that this unboundedness holds for all values of d.

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